THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

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THE MATHEMATICAL ASSOCIATION.

ANNUAL MEETING, 1912.

The Annual Meeting was held on Wednesday, 10th January, 1912, at the London Day Training College, Southampton Row, W.C., the President, PROFESSOR E. W. HOBSON, D.Sc., F.R.S., being in the chair.

The morning meeting began at 11 a.m., when the **President** delivered his address on "The Democratization of Mathematical Education."

Mr. C. Godfrey then spoke on the work of the International Commission on Mathematical Teaching; and **Mr. G. St. L. Carson** read a paper "On some unrealised possibilities in Mathematical education."

The afternoon meeting began at 2.30 p.m., when the Report of the Council for 1911 was read by the President.

The adoption of the report was proposed by Mr. T. J. Garstang, seconded by Mr. P. T. Stephenson, and carried.

The Treasurer's Report for the year 1911 was read and adopted.

The election of officers and Council for the year 1912 then took place. Miss M. Punnett (Vice-Principal of the London Day Training College) was proposed by Miss Hancock, seconded by Mr. A. W. Siddons, and duly elected as an Honorary Secretary in the place of Miss E. R. Gwatkin, who had resigned; Miss E. R. Gwatkin and Dr. F. S. Macaulay were elected members of the Council in place of Mr. W. J. Dobbs and Mr. C. Godfrey, who retired; and the other officers and members of the Council were re-elected.

A discussion on the question of the proposed scheme for the re-constitution of the Teaching Committee then took place, in

which Messrs. G. F. Daniell, A. W. Siddons, C. S. Jackson, G. St. L. Carson, H. D. Ellis, the President, A. G. Pickford, P. T. Stephenson, C. Godfrey, P. Abbott, and Dr. T. P. Nunn took part. The proposed scheme was eventually approved and adopted.

Mr. C. V. Durell read a paper on "A plea for the earlier introduction of the Calculus," and a discussion followed.

At 5 p.m. the Association was invited to an exhibition of scientific apparatus and books, arranged by the Association of Public School Science Masters.

In the evening the Association of Public School Science Masters and the Mathematical Association dined together at the Trocadero Restaurant. The dinner was followed by a very enjoyable entertainment.

THE MATHEMATICAL ASSOCIATION.

REPORT OF THE COUNCIL FOR 1911.

The Council are pleased to report that the Association continues to increase in numbers. During the year 1911, 87 new members have been elected, and the number now on the Roll is 675. Of these 9 are honorary members, 30 are life members by composition, 55 are life members under the old rule, and 590 are ordinary members. There are also 200 associates.

The Council regret to have to record the deaths during the year of two members of the Association: Professor G. Chrystal of Edinburgh, an honorary member, and Mr. A. C. Harris, Headmaster of the High Street L.C.C. School, Stoke Newington.

In connection with the Educational Section of the Fifth International Congress of Mathematicians to be held at Cambridge in 1912, this Association has undertaken to organise an exhibit of models, diagrams, appliances, books, etc., illustrative of mathematical education. In view of the fact that the International Congress is meeting in England for the first time, and that such an event cannot recur for many years, the Council trust that members of the Association will co-operate in an effort to make the exhibit worthy of the occasion and of the Association.

The various branches of the Association have displayed great activity. The meetings of the North Wales Branch have been well attended, and the discussions have turned upon the question of examinations and upon other matters connected with school work. At the meetings of the Southampton Society the principal subjects discussed have been "Infinities," the amended report of the Association on the teaching of Algebra and Trigonometry, and a beginner's course in Theoretical Elementary Plane Geometry.

The discussions were long, animated and helpful, and it is believed that the Branch is serving a very useful purpose.

The London Branch consists of about 150 members and 140 associates. Sir George Greenhill has been elected as its first President. Among the subjects discussed have been the teaching of Trigonometry, Ratio and Proportion, and Solid Geometry, "Field work in Elementary Geometry," "Pound and Poundal," "The Introduction of the Calculus," and "The place of Mathematics in the Education of Girls." The second annual soirée was held at the Polytechnic, Regent Street, in November.

Branches formed at Sydney, N.S.W., and at Toronto, have been recognised by the Council. The Sydney branch, of which Professor H. S. Carslaw is President, consists of 5 members and 37 associates.

A sub-committee of the Council has been appointed to consider the question of the formation of other branches in suitable centres.

The books forming the Library of the Association were transferred in July to the premises of the Teachers' Guild, 74 Gower Street, W.C., where they may be consulted by the members of the Association. A catalogue of the books is being prepared for publication, and when ready will be distributed to members. Regulations to control the borrowing of books from the Library have been drawn up, and will be issued with the catalogue.

A list of mathematical periodicals and of the places in London and elsewhere at which such periodicals may be consulted, is in course of preparation, and will be published in the *Mathematical Gazette*.

The Council have prepared a scheme for the re-constitution of the Committee appointed to consider improvements in the methods of teaching Mathematics, and copies of it have been sent to all the members and associates. The scheme will be laid before the annual meeting.

The Council regret that Miss Gwatkin has expressed a wish to resign her office as one of the honorary secretaries of the Association, and desire to express their cordial appreciation of the benefit which the Association has derived from her tenure of office, comparatively brief though the period of that tenure has been.

Mr. C. Godfrey and Mr. W. J. Dobbs now vacate their seats on the Council in compliance with the rules of the Association, and the members present at the annual meeting will be asked to nominate and elect their successors.

The Council desire again to express to Mr. Greenstreet their hearty appreciation of his work as editor of the *Mathematical Gazette*; and to the authorities of King's College and of the London Day Training College the warmest thanks of the Association for their kindness in affording accommodation for the many meetings which have been held in connection with its work.

THE MATHEMATICAL ASSOCIATION.

THE Annual Meeting of the Association, held at the London Day Training College, Wednesday, January 10th, 1912. PROFESSOR E. W. HOBSON, D.Sc., F.R.S., in the Chair.

VERBATIM REPORT.

1. THE PRESIDENTIAL ADDRESS.

THE DEMOCRATIZATION OF MATHEMATICAL EDUCATION.

THE work of the Mathematical Association, in connection with its activity in promoting the reform of Mathematical teaching in our schools, necessarily involves the expenditure of much time and thought upon the detailed discussion of specific schemes for the improvement of the teaching in special departments of Mathematical education. It is, however, well that we should sometimes reflect upon the more general aspects of our work; and perhaps a Presidential Address affords the most suitable occasion for reducing some such reflections to an explicit form, even though nothing essentially new can be said upon the matter. To attempt to offer opinions on many of the important questions of detail which come up before the Association and its Committees for discussion would be an impertinence on my part. Much of the experience requisite for the formation of considered opinions on points of detail is lacking to me; and I consequently regard myself as in the position of a learner in such matters. Indeed I consider myself fortunate in being allowed to hear the opinions of the many experts, of the greatest experience in all matters appertaining to the school teaching of Mathematics, who are to be found amongst the members of our Association.

In making a few brief remarks upon the general character of the reform movement, I propose to emphasize one or two governing principles which I regard as of fundamental importance in relation to Mathematical teaching. If I venture, in the course of my remarks, to make some suggestions on less general matters, the adoption of such suggestions as parts of the policy of the Association would only be possible after much detailed discussion of the manifold points which would have to reach some degree of settlement before the suggestions could be translated into the domain of practice.

The modern tendency which has exhibited itself in our time in greater or less degree in all countries, in educational policy in general, may be described as the tending towards the democratization of Education. This term, or some synonymous one, has frequently been used to denote the extension of Education to wider classes of the population; but it is not in this quite

general sense that I intend here to employ the expression. I mean by it rather the progressive adaptation of educational methods to the *intellectual democracy*; the transformation of the methods of teaching and of the matter of instruction so as to meet the needs of those who are lacking in exceptional capacity, at least in relation to the particular branch of study in question; in other words, the concentration of the attention of the Educator, in a much greater degree than formerly, on the work of developing the minds of the average many, and not solely of those of the exceptionally gifted few. The progress of democratization of Education, in this sense, has been perhaps more marked in the case of Mathematical instruction than in other departments. In our own country the Mathematical Association has been conspicuous as an agent in furthering the democratization of Mathematical Education. It is very certain that no such democratization could be effected without more or less radical changes being made both in the methods of teaching and in the selection of the matter taught. It would be of but little avail that the attention of the teacher should be concentrated in a greater degree than formerly on the average many, if the methods of teaching and the material taught remained unreformed. With a view to the formation of some estimate of the profit and loss due to the changes which have taken place of late years in the teaching of Mathematics in many of our schools, let me briefly glance at some of the differences, both in theory and in practice, which distinguish the older and the newer methods from one another. Any exaggeration of which I may be thought guilty must find its excuse in the fact that I am attempting to indicate only the more salient features in a continuously progressive movement.

In accordance with the older and traditional treatment of Mathematical instruction in our schools, Geometry was treated in a purely abstract manner; the idea being that Euclid, as a supposed model of purely deductive logic, should be studied entirely with a view to the development of the logical faculty. Any knowledge of space relations which might have been imparted by this study was reduced to a minimum by the excessive insistence on all the details of the syllogistic form; the whole attention of the pupils being engrossed by the effort to commit to memory a long chain of propositions in which the actual geometrical content was exceedingly small. On the other hand, Algebra, and to a great extent Arithmetic, were taught without any regard to their logical aspects, but mainly as affording discipline in the purely formal manipulation of symbols in accordance with prescribed rules; little or nothing being said as to the origin of such rules. The teaching of Mechanics was assimilated as far as possible to that of Geometry,

the true position of the subject as a fundamental part of Physical Science being almost wholly obscured. That the average boy or girl is not by nature appreciative of formal logic, or of the interest and meaning of abstract symbols, was thought to be a reason why the subjects so treated should be especially insisted on. In fact, the notion of Mathematical teaching was that it should be in the main medicinal and corrective. Its advantages consisted largely in calling forth the use of faculties which are the rarest in the average boy or girl, and were therefore thought to be in special need of development. It was thought to be by no means wholly a disadvantage that these subjects, so treated, were found hard and repulsive by the majority. It was thought that the hard discipline involved in the attempt to assimilate them developed a kind of mental grit, and involved a certain species of moral training, even when the intellectual results were small. A certain strengthening of faith, to be acquired in the process of hard work spent on subjects of which neither the aim nor the utility was obvious to the pupil, was thought to be highly beneficial.

It is unnecessary for me to enlarge upon the defects of this system, and on the inadequacy of the ideals underlying it. The existence of the Mathematical Association is a warrant of the widespread dissatisfaction with these methods, both in their results and their aims. The system as it existed in our schools was condemned by its failure. It failed to attain even its own narrow ideals, except in the case of a very select few among the pupils. The many rejected the material which was for them wholly indigestible mental food. The system was, in the sense in which I have used the term, undemocratic. The results obtained in the case of the vast majority were deplorable; and it needs indeed a strong faith in the anti-democratic principle to imagine that this failure was compensated by the effect of a hard and bracing training on the few who, by mental constitution, were enabled in some degree to profit by it. Even the chosen few suffered severely from the effects of the narrow conception of education which lay at the base of the methods of instruction. For the purely abstract treatment failed to disclose the close relations of Mathematical ideas with the physical experience in which the abstractions took their origin. That Euclid has any relation to the problems of actual space was seen by the majority of those who suffered under this system only at a later time, if at all. The relations of symbols with the concrete, and the economy of thought involved in their use, remained for the most part unappreciated; such appreciation came, if at all, as the product of later reflection on the part of a very few even of those who had attained to some facility in the manipulation of the symbols.

Mais, nous avons changé tout cela. The modern methods of teaching appeal in the first stages to those interests which are strongest in the majority, instead of running atilt against the most undeveloped sides of the minds of the pupils. Geometry, the science of spatial relations, is introduced by the observational and experimental study of the simplest spatial relations, verification by actual measurement playing an important part; the abstract treatment in accordance with the deductive method being relegated to a later stage. The interests of the average boy are rather practical than theoretical; therefore, it is thought, he must be interested with space relations on their practical side. He is not interested in formal logic, therefore he must not be bored with learning a chain of theorems of which the object is not apparent to him. He is not usually ingenious, therefore, it is thought, no demands must be made upon him which require ingenuity. He does not readily move in the region of abstract symbolism, therefore he must be introduced to the use of symbols only in an arithmetic manner, in which the concrete implications are prominent. Laborious exercises in Algebra, in which expertness in the manipulation of symbols is the object to be attained, should, it is thought, be for the most part omitted.

Owing in large measure to the activities of the Mathematical Association, a considerable transformation in the methods and in the spirit of Mathematical teaching has already taken place in many of our schools, and the changes in the direction indicated by the newer ideals are no doubt destined to have even more far-reaching effects than at present. However, the old mechanical methods of teaching still linger on in many of our schools, in which conservative traditions are notoriously difficult to eradicate. The detailed discussion, both in print and *viva voce*, which arise in connection with the work of our Association, may be of inestimable value in directing aright the detailed development of the reformed methods of teaching. I hope, also, they may prove useful in the direction of checking those one-sided exaggerations which are always apt to arise in connection with activities in which the objects to be attained are various; as they must be in the case of so many-sided a branch of education as the one with which we are concerned. Some degree of compromise, without undue sacrifice of principles, may often reasonably be made, in adapting the teaching so as to take account of the widely diverging future careers in prospect for different classes of pupils.

It may, I think, be safely maintained that, the better the theory underlying the method of instruction may be, the more exacting will be the demands made upon the skill, the knowledge, and the energy of the teacher. My own early recollections of learning Mathematics call up memories of the Classical Master,

without any real knowledge of, or real interest in, the subjects, hearing the repetition of propositions of Euclid, or setting a long row of sums in Algebra, monotonous in their sameness. Somehow, a few of us managed to learn something, but I tremble to think what would have been the result, had the said Classical Master attempted to teach in accordance with the newer methods. For the success of the teaching in accordance with the reformed methods, a high degree of efficiency on the part of the teacher is essential, if the results hoped for are to be attained; and even if those results are not in some respects to fall short of what was reached under the older system. The teacher must possess a high degree of skill in presenting his material; he must have a broad knowledge of the subject, reaching much beyond the range which he has directly to teach; he must have skill and alertness in handling a class, that skill having been developed by definite training, but of course presupposing a natural capacity for the kind of work. Some of the failures, of which one hears, of the newer methods to produce satisfactory results, may probably be traced to a falling short on the part of the teaching, in one or more of the points I have indicated.

At the present time it is not possible to form any precise estimate of the actual effects of the recent reforms in Mathematical teaching. It will only become possible to do so when the confusion incident to a state of transition has passed away. That in many quarters the gain has already been considerable, I have no doubt. Nor do I doubt that the principles underlying the newer methods are sounder than those which formerly held sway. I have no doubt that it is right to proceed from the practical and concrete side of the subject, rising only gradually to the more abstract and theoretical side. But the adoption of more correct principles is only one step; their actual translation into practice gives rise to many difficulties, and to many dangers, some of which have most certainly not been altogether avoided. The process of change has as yet not been one involving pure gain.

A perusal of some of the current treatises on "practical Mathematics" has led me to think that in some quarters the purely practical side of Mathematics is unduly emphasized. The teaching should, without doubt, commence with this side, and should never lose touch with it, but the study of Mathematics must be pronounced to be a relative failure as an educational instrument if it fails to rise beyond the purely practical aspect of the subject to the domain of principle. Purely numerical work, calculation with graphs, problems in which the data are taken from practical life—all these are excellent, up to a certain point, and they form the right avenue of introduction to scientific conceptions. But if this kind of work is unduly prolonged, and too exclusively

practised, it tends to develop a one-sided mechanical view of the capabilities of Mathematical methods, and the study ceases to be in any real sense educational. Such practical work is only educational when it precedes and leads up to a grasp of general principles, and when it is employed to illustrate such principles. I do not wish in the least to depreciate the importance of Mathematics as providing the tools for a vast variety of applications, useful in various professions. This side should never be lost sight of in school work. But the most important educational aspect of the subject is as an instrument for training boys and girls to think accurately and independently; and with this in view, the more general and theoretical parts of the subject should not be entirely sacrificed either to the exigency of providing useful tools for application in after life, or to the supposed need of sustaining interest in the subject by a too anxious adherence to its concrete and practical side.

I gather that, in some of the current teaching of practical Mathematics, a kind of perverse ingenuity is exhibited in evading all discussion of fundamental ideas, and in the elimination of reference to general principles. Instead of a skilful use being made of practical methods to lead up to general methods and illuminating ideas, practical rules seem sometimes to be made the end of all things. I have been told, for example, that the use of logarithms is sometimes taught to students who at no time attain to a comprehension of what a logarithm really is, or of the grounds upon which the rules for the use of logarithmic tables rest. Students who are in the habit of employing for purposes of calculation formulae the origin of which they do not understand have entered upon a path which will inevitably lead to disaster, not only as regards their mental culture, but also in the practical domain. If Mathematics is degraded to the level of a set of practical rules, of which the grounds are not understood, for dealing with practical problems of special types, the unscientific character of such a study will avenge itself even on the practical side of life. A student who proceeds on these lines will fail to arrive at those points of view that are not only the most stimulating mentally, but of which the attainment is really essential for success in applying Mathematics to practical matters. The practical applications of Mathematics are much too varied to be capable of being confined within the range of any number of prescribed rules and formulae. Practical problems will be found constantly to arise in connection with professional work which are not quite on the lines of the rules that have been taught, and these problems can be effectively dealt with only by persons who possess some real grasp of Mathematical principles, as distinct from a mere knowledge of certain practical rules and methods. Whilst maintaining that a student should thoroughly understand

the grounds upon which the formulae and rules which he employs are based, I do not believe that he ought to be expected to commit to memory, and to be able to reproduce at any time, formal proofs of all such formulae and rules. Much precious time and energy has been unprofitably employed in the past in attempting to satisfy the unreasonable demands made by Examiners in some branches of Mathematics that formal proofs should be forthcoming of everything that the candidates are supposed to have learned. The burden thus thrown on the memories of the candidates is far too heavy; and much time and energy which should have been employed in an endeavour to grasp and realize principles has thus been diverted to a far less profitable use.

It appears to me to be eminently desirable that the time saved by the diminution, in school work, of the amount of time spent on unessential details and on unnecessarily prolonged drill in the manipulation of symbols should be employed in introducing the pupils to a considerably greater range of Mathematical thinking than has hitherto been usual; and in particular in endeavouring to make them acquainted with more of the fundamental and fruitful ideas which make Mathematical Science what it is. In the higher classes some time might profitably be spent on the principles, as distinct from the practice, of Arithmetic. It would be of great educational value if the principles which underlie the practice with which all the pupils have become familiar were brought explicitly to their consciousness. For example, they should understand the principle of our arithmetic notation, so that they may have an adequate appreciation of its beautiful simplicity, and of the fact that it embodies a great time-saving invention. In order to attain this object it is necessary to deal with the theory of scales of notation and radix-fractions, so that the arbitrary element involved in the adoption of the scale of ten may be clearly appreciated. I do not of course contemplate the introduction into such a course of artificial problems on scales of notation; only the fundamental principles should be explained, with such quite simple illustrations as may be found necessary for their complete elucidation.

I do not know to what extent some rudimentary and informal treatment of the properties of simple figures in three-dimensional space has at the present time become part of the normal instruction in Geometry in our schools. I am quite sure of the urgent necessity for finding time for a small modicum of study of this part of Geometry. I remember, a few years ago, in a paper on Mathematics for Candidates for a College Scholarship in Physics, the candidates were asked to construct the shortest distance between two given non-intersecting straight lines. One of the candidates, who showed a considerable knowledge of plane

Geometry, informed me that two non-intersecting straight lines are necessarily parallels. It is unnecessary to insist upon the importance of an endeavour to uproot ignorance of this kind, due as it is to lack of stimulation of the power of observing simple spatial properties.

In considering the various directions in which Mathematical teaching may be made to extend beyond the domain that consists of that drill in the employment of processes which up to a certain point is undoubtedly necessary, one question of great importance arises. That is the very important question as to the possibility of making a rudimentary treatment of the ideas and processes of the Calculus part of the normal course of Mathematics in the higher classes of schools. In the hands of a really skilful teacher, the purely formal element in the treatment of the Calculus could be reduced to very small dimensions; all the leading notions and processes could be sufficiently illustrated by means of functions of the very simplest types. I believe that some of the time saved by lightening the matter in such subjects as Algebra might be more profitably employed in this manner than in any other. The Calculus, as embodying and utilizing the fundamental notion of a "limit," is the gate to a Mathematical world of incomparably greater dimensions than the one in which the student has moved during the earlier part of his course. Any method of presentment which evades the notion of a "limit," as it appears in the differential coefficient, or in kinematics as a "velocity," is much to be deprecated. The possession of this notion is the most valuable result of the study, both for educational and for practical purposes. By means of carefully chosen examples, in both the arithmetic and the geometric domains, a pupil may be led up to this fundamental notion, so that it may in the end become really his own. To this end it is wholly unnecessary that any treatment of the subject should be employed which would satisfy the logician or the professional mathematician. The important point in connection with this idea, as with many others, is that the student should really have the notion as part of his permanent mental furniture; and not that he should be able to give a complete description of it, or of its philosophy, in conceptual language. I do not propose to indicate now, even in outline, a schedule of those parts of the Calculus which would be suitable as part of a general education. This is a matter which might with much advantage be fully discussed by the Association, when the views of practical teachers as to the possibilities in this direction would receive the fullest attention.

There is a danger which arises in connection with the democratization of education, that less than justice may be done to the minority, who, by natural aptitude, are capable of making much more rapid progress than the rank and file. This danger is

probably not so great in this country as in some others; with us, the old leaven which impels teachers to make the most of their more gifted pupils still works strongly enough, and the questionable stimulus provided by Scholarship Examinations and other competitions exercises an influence in the same direction which is very powerful, and perhaps indeed too powerful. In some countries the rigid system by which every pupil in a school is taken in a general class, in a certain number of years, through prescribed portions of a subject, acts detrimentally upon those pupils who are capable of learning much more rapidly than the average. In America, I was told that it would be regarded as undemocratic to make any special provision in a school for the more rapid advance of gifted pupils. This view seems about as reasonable as it would be to prescribe, as a thoroughly democratic arrangement, that all the pupils should be supplied with boots of the same size. The general good demands that, as far as possible, equality of opportunity should be afforded to all for their mental development in accordance with their enormously varying abilities; it does not demand a mechanical equality of treatment represented by forcing all students to move at the pace of the less gifted or of the average. Although, however, this danger may be a real one in some quarters in this country, the opposite fault, of sacrificing to some extent the needs of the average to those of the abler students, is probably still the more prevalent one.

The movement which I have spoken of as the Democratization of Mathematical Education is a progressive development. Something not inconsiderable has been accomplished in our time; very much more remains to be done. The difficulties which arise in this connection are largely those of finding the true coordination between the practical and the theoretical sides of the subject. An undue emphasis placed on either side is apt to have disastrous results. The perfect mean is in all such cases probably an unattainable ideal; a certain degree of compromise, depending upon a variety of circumstances, is usually the practicable course; but the most earnest endeavours should be made to prevent such compromise going too far. Whilst recognizing to the full the importance of the practical side of Mathematics, both as affording the right approach to the subject, in view of sound psychological principles, and also on account of its importance as an equipment for various departments of practical life, let us never lose sight of the paramount importance of Mathematics as part of a real education of the intellect. Such education is incomplete unless a few at least of the many illuminating notions which our race has achieved in its long struggle to attain clearness in the domain of Mathematical thinking are made the common property of our *intellectual democracy*.

ON THE PRESIDENT'S ADDRESS.

Mr. Daniell said: I am sure we are all very grateful to our Chairman for his address. There is one question which I should like to ask. Our Chairman used the term "Practical Mathematics." Now, the word "practical" is unfortunately employed in regard to mathematics in two totally different senses. The Board of Education and Professor Perry speak of practical mathematics in the sense of occupational mathematics—mathematics that shall train learners for their subsequent vocations. The expression is used also in another sense, I mean in its connection with work in the laboratory. I do not feel sure which application was intended in the address.

The Chairman: I think I had both in my mind. I used the word "practical" to distinguish from the mathematics which deals with principles.

2. ON THE WORK OF THE INTERNATIONAL COMMISSION ON MATHEMATICAL TEACHING.

By Mr. C. Godfrey, M.V.O.

The few remarks that I propose to make on this subject will hardly deserve the title of a "paper." You are doubtless aware of the existence of the International Commission on Mathematical Education. The Commission owes its origin to the distinguished American mathematician, Professor D. E. Smith. At the International Mathematical Congress at Rome three years ago Professor Smith proposed that a "commission" be formed to enquire into questions of teaching, this Commission to report to the next meeting of the Congress, at Cambridge, on August 22-28, 1912. There have been intermediate meetings on the teaching part of the subject, and one meeting was held last September at Milan. It was not a general meeting, being attended mainly by official delegates from different countries. I had the honour of being present there, and it may perhaps interest you if I describe very briefly the matters that were under discussion.

The first question was this. It was desired to ascertain the existing state of things in the different countries of the world as to the degree of rigour that the teachers expect and wish to develop in the treatment of mathematical problems; the degree of rigour on the one hand, and on the other hand how far methods of intuition are allowed to play their part—rigour *versus* intuition. The meeting did not discuss what ought to be the practice; it wanted to find out what is the practice.

As regards what is being done, a rather interesting result arose from the discussion. It was found that, of the countries represented by the delegates, the country which makes most demands for rigour of treatment is Italy. In Italy school teaching has been to a very great extent controlled by university professors. Apparently the influence of university professors has been cast in the direction of rigour of treatment, the methods adopted not being at all of the kind which our Chairman has suggested to-day. And I gather that at the present moment they are wondering whether they are on the right track.

I should explain that the discussion naturally tended to focus itself on geometry, and when one speaks of rigorous treatment of geometry an Englishman naturally thinks that Euclid is meant. Now, that is not the case in Italy. The Continental mathematicians do not regard Euclid as rigorous. When they speak of rigour they have in mind work on the lines of Hilbert and Peano. In their treatment it is necessary to discuss such points as what exactly is meant by "between." The word "between" must be explained by a series of definitions before you are allowed to use it. Again, with regard to "equal" you must have a definition of equality that does not involve superposition, because superposition involves taking a length from one place to another place, and you really cannot tell that it has not altered on the way. This conception of rigour, demanded by Italian methods, has apparently in theory been imposed upon Italian schoolboys.

Well, at the other end of the scale stands Germany. In Germany the freest use is made of intuition, and the sphere for rigorous mathematics is admitted to be the university. We find Italy at one end of the scale and Germany at the other. It appeared from the discussion that France and England occupy an intermediate position, France leaning somewhat towards the Italian treatment, and England somewhat towards the German treatment. The American representatives were not present and their views were not obtained.

In the discussion one was very conscious that on the Continent mathematical criticism has reached a more developed stage than in England—criticism on the methods of teaching, and so forth. What I mean is this. They have thought out various possibilities, and they have given names to their results. For instance, the question of "Fusion" *versus* "Purism." I never heard of fusion or purism before; and I take it that in this country mathematical criticism has not yet reached the stage of development in which those words can be used. Fusion is the attempt to combine the teaching of various subjects; purism is the ideal which says that each subject must be taught without the introduction of another. For instance, the fusionist school would

combine analytical and geometrical conics, trigonometry and geometry, the integral and differential calculus; the purist school would separate them.

The third subject discussed was to what extent schools and universities provide special courses of mathematics designed to apply to subsequent occupations in life. Under such a heading one naturally thought of the question of engineering, but that was ruled out of the discussion as being too large a matter. Consideration was confined to the extent to which mathematical courses are supplied in schools and at universities for students of biology, statistics, insurance, problems of administration, and so forth. In Germany, of course, administration is a subject of considerable study. One was surprised to hear a remark from Professor Klein that in Germany the watchword of university education was liberty. Each university does exactly what seems fit in its own eyes, which means that they all do different things, and it is almost impossible to ascertain the state of things in university education on any given point.

The most interesting point that emerged from the discussion was the experience of France. In France, at one time, it was attempted to provide special courses in the universities to cover the various applications in after life. On experience, those courses have been abandoned in favour of courses at schools providing for those possibilities. And that is done as follows:— The unessential has been eliminated from the school course, and the time saved in that way has been used to give a wider foundation to a boy's mathematical knowledge. Such wider development as has been suggested by our Chairman now forms part of the education of the average boy at school, and it is found that this widening of the mathematical scheme at the schools provides a better solution for the problem than anything in the same way at the universities.

No doubt you are aware that a considerable amount of literature has been got together by the secretaries of the sub-committees in the different countries. The meeting at Cambridge will provide an opportunity for students of mathematical education that may not recur. I understand that anyone may attend on payment of a guinea. To my knowledge there has never been anything quite on the same scale before. It may interest members to learn that the French reports are all published and can be bought.[1] The reports describe mathematical education in the universities, schools, and other institutions. The English reports are being prepared, and I have ready to circulate a list of them,[2] though Mr. Jackson, the Secretary of the English

[1] *Rapports de la Commission Internationale de l'Enseignement Mathématique.* Hachette & Co., Paris and London.

[2] Wyman & Sons, Fetter Lane, E.C.

Committee, tells me that one or two of the papers are rather "in the air." The German reports are published by Teubner, Leipzig. Information as to any of the publications of the Committee may be had from M. Fehr, 110 route de Florissant, Geneva.

I think I have now discharged my promise to the Council to make a few remarks on this subject. (Applause.)

The Chairman, in inviting discussion, said: We sometimes hear that people like to go back to Euclid as a text-book of geometry. Euclid is not regarded as rigorous by either school. It is not rigorous and it is not practical, so that it falls between two stools altogether. It satisfies neither the demands of perfect logic nor of practical needs.

Dr. Macaulay: I do not quite understand what Mr. Godfrey said about France. He seemed to say that France had found a method of quickening and vitalising school mathematics by teaching special branches, such as statistics, as a preparation for after life; but later I thought he said that they had succeeded in a great measure in leaving out the unessential in mathematics and broadening the basis. These statements do not seem to mean quite the same thing. I did not understand whether the object in France was to make special preparation for the after occupations of life, or whether they tried to broaden the field of mathematics so that boys might have a better foundation for profiting by university teaching.

Mr. Godfrey: As far as I can make out, in France they find that the best way of making mathematics available for subsequent life is to broaden the basis in the way that Dr. Macaulay mentioned. They think that a narrow curriculum for Mathematics cannot be a basis for anything satisfactory, and that the teaching must be broadened by the addition of the calculus, trigonometry, and so forth, and that then it does become valuable for superstructures in after times. That is, I believe, what is done in France. I do not at all mean to say that special courses in insurance, for instance, are given there in the schools.

3. ON SOME UNREALISED POSSIBILITIES IN MATHEMATICAL EDUCATION.

By Mr. G. St. L. Carson.

THE last half-century has seen a great and significant change in the popular estimation of mathematics. Formerly the subject was regarded as utterly unpractical, and therefore useless in the narrow sense of this term, though it was recognised as providing a training, unique in its character, in logical thought and accurate

expression. Now it is regarded, and correctly regarded, as having enormous practical importance in science and engineering. Most, if not all, of those discoveries and inventions which are so profoundly modifying civic and national life have found their origin, or development, or both, in the labours of mathematicians, and this fact is widely known. The mathematician is no longer regarded as a dreamer of dreams; he is classed with the doctor, the engineer, the chemist and all those whose specialised labours have had immense import for the human race.

But simultaneously a change of no less magnitude has taken place in the mathematical world. The type of investigation which bore such fruit in the hands of Faraday, Clerk Maxwell, Kelvin and many others no longer occupies the attention of those who are in the forefront of mathematical investigation. The theories of pure number, of space, of functions, and such names as Dedekind, Cantor, Grassmann, Klein, and, in our own country, Hobson, Whitehead and Russell, have little or no connotation for the outer world. In so far as this outer world is cognisant of their existence, these theories, and the men to whom they are due, appear as chimerical and unpractical as would the labours of Clerk Maxwell have appeared to the Lancashire cotton spinner of 1850. And I fear that this view is too often shared, consciously or unconsciously, by mathematicians themselves, and especially by those who teach the subject. Here, they say or think, is a type of thought or investigation of great interest to those who can appreciate it, but it is utterly and permanently out of touch with the world at large. It can have no relevance or import for the ordinary boys and girls who learn mathematics at school, and can in no way assist them to become efficient workers and citizens.

But is this really the case? He would be a bold man who would say with certainty that any branch of scientific investigation must be regarded, once for all, as having no bearing on the development of the individual or race. Is not the better answer that the practical import of these investigations has not yet been perceived; that it behoves all mathematicians, but especially those who are engaged in teaching, and therefore have some knowledge of the youthful mind, to do what they can to correlate this work with the outer world, and to examine to what extent it can now influence the manner or matter of teaching in our schools? The question will probably receive an affirmative answer from each one of you, but you may perhaps add that I am walking in the mists which hide from us the development of future centuries; that sufficient unto the day is the vision thereof; and that the ground to which I invite you is a morass which may conceivably be made firm by our great-grandchildren.

Nevertheless, I am going to ask you to bear with me while I

endeavour to convince you that we can now commence to bridge the morass. I admit that it is one. Hesitating and imperfect our endeavours may be, but I am honestly convinced that the time is ripe for a commencement, and that the future of mathematics as a universal subject in the curricula of schools depends, in some part at least, on this commencement being made at once. My ground for this conviction is best stated tersely. I believe that the modern theories of pure mathematics are destined to illumine our understanding of the human mind and of cities and nations, just as the pure mathematics of fifty years ago has already illumined the previously dark and chaotic field of physical science; that modern mathematics is or will be to psychology, history, sociology and economics as has been the older mathematics to electricity, heat, light and other branches of physical science. For example, it may well be that the theory of sets of points or the theory of groups will find fruitful application in economics. You will see that I am suggesting that the range of applied mathematics may be widened far beyond its present scope. It was asserted yesterday at a meeting of headmasters that the reign of pure mathematics was closed. Would it not be more accurate to say that pure mathematics has of late extended and coordinated its dominions to an amazing extent, and that corresponding extensions of applied mathematics have yet to be found? If I am right in this, and I admit that the speculation is a daring one, our subject has an irresistible claim. We may trust our lives to engineers and scientists just as we entrust our bodies to doctors and surgeons; but each member of a human society should, so far as he may, be competent to analyse and estimate for himself the workings of his own mind, and the development of the society of which he is a unit. In the more detailed remarks which I am about to make, I will ask you to bear in mind that their main inspiration and justification lies in what I have just said; that they represent an individual attempt to relate mathematical education to human thought and social development.

Mathematics has been defined by Russell as the class of propositions, "If A, then B," and is applied to classes of entities concerning which certain propositions A are assumed; the truth of these is no concern of the subject. The entities form the universe of discourse. They can be ordered in respect of each of the attributes which characterise their class. This universe of discourse may be of any number of dimensions from one upwards; in Arithmetic it is one dimensional, and in Geometry it should be three dimensional, but is more often two dimensional. I may remark in passing that some attempt to estimate the number of dimensions, *i.e.* of quantities required for exact specification, of the entities discussed in such subjects as economics would often

throw considerable light on these subjects. The abstract idea of entities and their dimensions is too often wanting. Hence Arithmetic forms the basis of Mathematics, since it explores the properties of one-dimensional fields. Any treatment of Arithmetic which fails to explore the whole domain of such fields is *ipso facto* incomplete, and its victim is in possession of an imperfect instrument which cripples him alike in concrete and abstract applications. My first plea is, therefore, for a mathematical treatment of Arithmetic from the earliest stages.

There is much which might be said concerning integers and fractions, and in particular scales of notation. My omission of these subjects is only to be interpreted as an admission that decimals and the theory of exact measurement are of more immediate importance, and must occupy such time as I can devote to Arithmetic. To my thinking, young children are hurried on to fractions far too soon. There are many unexplored fields of concrete problems, possessing real interest for young pupils, study of which would give a much firmer basis for future developments than is now obtained. And the proofs of such simple rules as "casting out the nines" may provide easy exercises in deduction, not without value.

To commence, then, with measurement. When, in actual practice, one measures a length, there are three distinct objects, any one of which may be in view. The purpose may be either (1) to state a length greater than that of the given object, but as little greater as may be, or (2) to state a length less than that of the given object, but as little less as may be, or (3) to state two lengths as close together as may be, between which the given object lies. I venture to suggest that training in measurement can only become of any value (other than manipulative) if it proceeds on these lines, phrases such as "nearly" and "exactly" being abolished as inexact, and therefore unscientific. "Nearly" is useless until we are told *how* near or within what nearness, and "exactly" only means "as nearly as I can see." By the use of a vernier—the theory of which should be included in every course of Arithmetic—children should learn how nearly they *can* see, and then say, *e.g.* 13·4 cm. within ·2 mm. We should thus sweep away all the loose statements which are, I honestly believe, responsible for much of that lack of accurate thought which is the subject of present complaint, and replace them by a training in the exact expression of practical measurements, the final form being of the type "between 7·38 and 7·39 cm."

The ground is now prepared for the extension of the idea of number, this being done, probably, in connection with mensuration, *i.e.* by questions such as "Find the length of the side of a square whose area is 2 sq. in." Few trials are necessary in order to ensure conviction of the fact that the number of inches is not

a fraction, and systematic approximation from above and below is attempted. By actual trial (using multiplication only) it is found that the following pairs of numbers are respectively smaller and greater than the number required: (1, 2), (1·4, 1·5), (1·41, 1·42), and so on. So far nothing more appears than can be realised by measurement, but it is at once seen that (1) this process can be continued indefinitely, given time and energy, and (2) that there is no limit to the closeness of the approximation. The human mind, by this systematic approach, has thus ridden rough-shod over the imperfections of physical measurement. The latter leaves, and must always leave, an unexplored gap which cannot be diminished, but the method of successive approximation enables us to diminish the gap below any limit, however small.

Now this process, if carefully developed, is not beyond the comprehension of young pupils, and it may fairly be said to contain the germ of any proper study of functions and the calculus, whether this be undertaken on a graphical or analytical basis. In either case this method of inclusion between converging pairs is essential to any exact comprehension of the subject. And beyond this it develops the theory of pure number so far as to give the pupils—however unconsciously—an early example of a perfect mental structure, fashioned by extension from concrete experience, and it gives them the only true ideal for the exact estimation of any set of phenomena.

Shortly, then, I suggest the continuous development of the idea of a cut, a *Schnitt*, of the rational numbers, commencing it at an early age in connection with a scientific treatment of simple measurements, the purpose being to give a true concept of number in its relation to measurement.

I next make some reference to Algebra, stating first that I am not to be taken as implying that the subject should be taught before Geometry. On the contrary, I am convinced from actual experience that Geometry should have been studied for two years at least before Algebra is commenced.

At the risk of appearing to raise needlessly large issues, I must ask the question—what is an Algebra for our present purpose, and what educational purpose may be served by its study? To my mind there are two essential steps in the development of an Algebra; the first is the development of a symbolism which is (usually) suggested by certain combinations of entities—*e.g.* $a+b=b+a$, $ab=ba$—and the second is the extension of this symbolism to cases which bear no interpretation in terms of these entities, and its subsequent application to other classes of entities. By this I mean interpretation of symbols such as $3-5$, $x^{\frac{1}{2}}$, $3+\sqrt{-7}$, in each of which the entities originally considered are found to form part of a larger class. I propose to allude shortly to each of these steps.

As regards the first I have little to say, for the unrealised possibilities with which I am concerned are here not conspicuous. But I do feel that the laws of Algebra have received far too little attention in current and past teaching, in that their interpretation is so exclusively confined to the domain of pure number. Any ordinary boy or girl of 15 is able to realise that $a+b=b+a$ and $a+(b+c)=a+b+c$ are true when a, b, c are vectors, and to make simple deductions therefrom as, for example, the proof of the median properties of a triangle. Such work, even if only a little time be devoted to it, gives a larger and truer view of Algebra as a language with more than one interpretation. And it gives the idea of an Algebra relevant to any field of human thought, an idea far more stimulating and fruitful for the ordinary man or woman than the narrower view of one absolute Algebra, which is too often the only result of our teaching. But, when all is said and done, this first part of the subject only presents itself as the formation on methodical lines of a shorthand language; every step in the solution of equations, factorisation or what you will, can be expressed in words whether the entities be numbers or vectors, and no new methods are involved.

But now take the second step, the interpretation of algebraic symbols such as $x^{\frac{p}{q}}$ or $\sqrt{-7}$, which have at first no meaning. The process involved is (or should be) purely logical. We assume the same laws of combination (*e.g.* $x^m \times x^n = x^{m+n}$, $(x^m)^n = x^{mn}$) as are known to hold in cases which already bear interpretation, and then find that the one interpretation $x^{\frac{p}{q}} = \sqrt[q]{x^p}$ is consistent with each of these laws. It is too often assumed without proof that, because the one law $x^m \times x^n = x^{m+n}$ leads to this interpretation, the other laws, such as $(x^m)^n = x^{mn}$, must also be true in this case. I do not believe that complex exercises in the manipulation of fractional and negative indices can be of any profit, but I am convinced that a complete and logical interpretation of these indices, if only in particular numerical cases, can and should form part of every course in Algebra. It is one of the best examples of constructive logic to be found in elementary mathematics, and it gives a sense of new methods for the discovery of hidden fields of entities which is hardly to be found elsewhere.

Passing now to imaginary expressions, I would suggest that the geometrical interpretation of these is not beyond the capacity of pupils of 17 or 18 years of age, and further, that it provides a valuable link between the symbolism thus far developed and geometry of two dimensions. Not much knowledge of Trigonometry is required in order to understand the meaning of the expressions $a+bi$, $(a+bi)(c+di)$, nor is it necessary to plunge into

useless elaborations. The pupils have ample scope for exercise in written descriptions of the processes and arguments; for example, in showing that this interpretation satisfies the laws $z_1+z_2=z_2+z_1$, $z_1(z_2+z_3)=z_1z_2+z_1z_3$. Work of this kind provides excellent material for short essays, a side of mathematical work which has received scant recognition, if any. The power of logical thought is a poor thing if its possessor is incapable of clear expression of his ideas, and this type of essay writing is well calculated to stimulate expression.

At this point the pupil may well review his experience of Algebra. One after another apparent impossibilities of interpretation have been surmounted. Is there an end to the process, or can we go on in this manner indefinitely? The answer is, of course, that the performance of any algebraic operation on a quantity of the type $a+bi$ produces another quantity of the same type, and the process is closed. I would suggest that there is no inherent difficulty in the proof of this, granted a knowledge of elementary trigonometry, and that the view of Algebra so gained is of real value as showing that the exploration of the field of entities under discussion has been completed.[1] If the boys and girls of the future can reach this point, they may, I admit, forget, and rightly forget, many of the details of their education, but this idea of the exploration of a field of entities, and the demonstration that this exploration is complete may remain with them. If this be so, I do not think that you will question its value in dealing with the problems which present themselves —or should present themselves—to every citizen of a modern state.

Finally, I must make some reference to Geometry. The primary value of the subject is, in my opinion at least, that it develops a power of dealing logically with manifolds of two and three dimensions. When we prove that, if A and B are fixed points, and the point P moves so that the angle APB is constant, then P must lie on one of two arcs of circles, we are selecting from all the points of the plane those which enjoy a certain property, and showing that a certain other property is a necessary consequence of this principle of selection. And we develop the consequences in order to encourage the dormant faculty of selecting some set of a class of entities (the points in the plane) and examining their properties, not by the imperfect method of measurement, but with the relentless certainty of logical reasoning. But I will not dilate further on this aspect of Geometry, as it can hardly be called an unrealised possibility of education.

[1] The entities are typified by the points of a plane, denoted by symbols such as 2, $-\frac{1}{2}$, $\sqrt{3}$, $7+4i$, and the exploration is not carried to three dimensions, as might have been expected after the extension from a line to a plane.

My first suggestion in regard to Geometry is that some simple idea of methods of transformation, such as projection and inversion, should form part of every course, at any rate for pupils who continue the subject until they are 18 or 19 years of age. Such transformations contain the idea, not illustrated so completely elsewhere in elementary mathematics, of a correlation between two sets of entities, such that to each entity of one set corresponds a definite entity of the other set; and from the known properties of one set we derive properties of the other set. It may, I know, be said that the study of graphs involves this idea, but they deal with one-dimensional sets only, and a general idea cannot be gained by one illustration. The problems which concern the ordinary citizen must often involve sets of entities of several dimensions, and if he has attained to some idea of the correlation of such sets, and the examination of a new set in the light of known properties of an older set, he must thereby have more likelihood of forming some definite conclusions instead of floundering in vague uncertainties.

My last suggestion—and perhaps the most startling at first sight—is that older pupils should be given some idea of the nature of non-Euclidean Geometry. One of the most vicious fallacies with which we are encumbered is the idea that our postulates of space—and in particular the parallel postulate—possess an absolute certainty which is denied to every other statement which is the result of experience. Most of us regard the parallel postulate as more obvious and certain than, say, the statement that all men must die some day, and we are utterly wrong in so doing. An outline of the idea and history of non-Euclidean Geometries—I would refer especially to Poincaré's illustration and the recent paper by Carslaw[1]—is sufficient to dispel the idea, and to exhibit our space postulates as mere assumptions which fit our experience more simply and nearly than any others which can be made. I am not speaking at random; I have aroused keen interest and considerable comprehension in a form of classical specialists whose knowledge of Geometry was distinctly limited. To what end, you may ask. In showing the true relation between thought and experience, the manner in which the mind deals with the sensations which reach it from the outer world. Far as we have progressed, the saying "Man, know thyself" still has force. No experience with which I am acquainted shows so conclusively the relation of each of us to the universe as the discovery that the supposed certainties of space are pure assumptions; as much so as Newton's laws of gravitation and motion, or Darwin's theory of evolution.

You may, I fear, regard me as an unpractical visionary who

[1] See *Fundamental Concepts of Algebra and Geometry*, by J. W. A. Young. Also *Proceedings Edinburgh Mathematical Society*, vol. xxviii., article by Carslaw.

has put before you a host of ludicrously impossible suggestions. But I would ask you to stop and consider whether they really are impossible, and I would remind you that I have suggested nothing that has not been attempted, in outline at any rate, with ordinary pupils, and with some measure of success. And I would ask you to remember one thing more. The whole world is going through a transformation, due in part to scientific and mechanical invention and in part to the growth of separate nations each with its own methods and ideals, of which no man can see the outcome. Our function, the function of all teachers, is to produce men and women competent to appreciate these changes and to take their part in guiding them so far as may be possible. Mathematical thought is one fundamental equipment for this purpose, but mathematical teaching has not hitherto been devoted to it, because the need has but recently arisen. But now that it has arisen and is appreciated, we must meet it or sink, and sink deservedly. Neither the arid formalism of older days nor—I say it in no spirit of disrespect—the workshop reckoning introduced of late will save us. The only hope lies in grasping that inner spirit of mathematics which has in recent years simplified and coordinated the whole structure of mathematical thought, and relating this spirit to the complex entities and laws of modern civilisation. Even though every suggestion that I have made be fallacious and impossible, this one statement remains, and the future lies with those who first achieve success in directing mathematical education to this end.

THE DISCUSSION.

The Chairman: I am sure we have all listened with very great interest to Mr. Carson's very valuable remarks. There will be no two opinions about that, though there may possibly be some difference of opinion as to the practicability of some of his proposals. It struck me that he gave a very simple illustration of the way in which an irrational number may be dealt with, and I was very interested in one of the remarks at the beginning of his paper. He said that the man-in-the-street no longer regards mathematicians as dreamers of dreams. I am inclined to think that if by mathematicians he means the few really great mathematicians, the man-in-the-street is very wrong —(laughter)—because no great intellectual work is done in the world, or ever has been or will be done, except by dreamers of dreams. (Applause.) But all the same, I appreciate quite well what Mr. Carson means—that the man-in-the-street knows that mathematics in some form or another is absolutely essential to scientific work of the most practical character, and I dare say

the man-in-the-street knows that if it had not been for the labours of the mathematicians among others, we should have no taxi-cabs to-day. If he doesn't know it, then he ought to.

The Chairman having invited discussion,

Dr. T. P. Nunn said: We shall all agree with you, sir, in welcoming Mr. Carson's valuable and interesting paper. I should like, by a personal acknowledgment, to emphasise his point that teachers can gain immensely from the study of the modern logic of mathematics. During recent years I have entirely reformed my own earlier notions on the teaching of elementary mathematics, largely under the influence of such works as those of Mr. Bertrand Russell and Dr. Whitehead. The teaching of positive and negative numbers is a specific instance. Formerly, like the writers of most of the current text-books, I based my treatment of this topic upon the distinction between greater and less—teaching the absurd doctrine that negative numbers are less than nothing. The slightest knowledge of the logical doctrine of order is sufficient to make this unwholesome practice impossible. There is no other corner of the elementary teaching of mathematics which is not also profoundly affected by the study of modern logical notions.

I am bound, however, to add that I differ from Mr. Carson on the question of pedagogical procedure. He tells us rather emphatically that extensions of notation beyond the scope of the original definition should be adopted as the result of an abstract logical inquiry. For example, having taught our pupils the definition and properties of x^n where n is a positive integer, we are, he says, to raise, in a purely logical spirit, the question whether a meaning can be given to the notation when n is a fraction or negative. History—which so often gives us valuable indications of the best way of approaching such questions—suggests a very different mode of procedure. Those who are acquainted with the work of John Wallis will remember that he invented negative and fractional indices in the course of an investigation into methods of evaluating areas, etc. He had discovered that if the ordinates of a curve follow the law $y=kx^n$, its area follows the law $A=\frac{1}{n+1}.kx^{n+1}$, n being (necessarily) a positive integer. This law is so remarkably simple and so powerful as a method that Wallis was prompted to inquire whether cases in which the ordinates follow such laws as $y=k/x^n$, $y=k\sqrt[n]{x}$ could not also be brought within its scope. He found that this extension of the law would be possible if k/x^n could be written kx^{-n} and $k\sqrt[n]{x}$ as $kx^{\frac{1}{n}}$. From this, from numerous other historical instances, and from general psychological observation, I draw the conclusion that extensions of

notation should be taught because and when they are needed for the attainment of some practical purpose, and that logical criticism should come *after* the suggestion of an extension to assure us of its validity.

It will, I hope, be understood that I am supporting Mr. Carson's demand for clear thinking in the teaching of mathematics. I differ from him not with regard to its importance, but solely with regard to its position and functions in elementary instruction. I express this difference of opinion chiefly lest some among us, offended by a one-sided insistence upon the logical aspect of mathematics, should underestimate the importance of his otherwise excellent paper. (Applause.)

Mr. Godfrey: I suppose Mr. Carson would admit that there is a time when each boy and girl becomes ripe for the consideration of these somewhat philosophical issues in mathematics, and that before the time of ripeness is attained it would probably be a mistake to force these considerations upon them. It is altogether a question of when this time is, and it is very difficult to say when the time of ripening does arrive. We can only judge from our own experience, and I must admit that in my own case the time came rather late. I remember it quite well. It was, I think, after I had taken my degree. Perhaps the confession is rather shameful, but I then began to have doubts as to whether I could really prove that $\sqrt{2}\times\sqrt{3}=\sqrt{6}$. I began, I mean, to have doubts whether the proof I had accepted up to that time was satisfactory. Of course it was not. I had never been able before to multiply together any commensurables. It was interesting to go back upon all that, and very stimulating, and I remember feeling at the time how very wrong my mathematical master had been to leave me in ignorance of all these things. (Laughter.) But riper experience has convinced me that he was quite right, and that I should have been very bored if he had taught them to me at school. I may have been very slow in developing; but the whole question is, What is the state of development at which these discussions become appropriate?

Mr. Daniell: I should like to mention a personal experience in this matter of philosophical treatment. I had a boy sent to me who had a rather peculiar history in regard to mathematical studies. He had won a prize for mathematics low down in a big public school. When he came to me at the age of seventeen, I found that he knew no mathematics at all. That is why he was sent to me. (Laughter.) The boy said, "I do not understand how it is you can multiply $\frac{1}{2}$ by $\frac{1}{3}$." I think that was a very sensible difficulty. It was not the case that he could not deal with the operation; he simply did not understand the philosophy of it, and it bothered him for a long time. It was just possible that this boy, reputed stupid in mathe-

matics, was not stupid at all. In fact, there were many things outside school-work which showed that he was thoroughly intelligent. But mathematical study was absolutely barred him by the philosophical difficulties not being faced. Possibly his teacher may not have known how to face them. It is most important that teachers, at any rate, should have studied these philosophical questions, and be able to deal with the difficulties of their pupils.

Mr. Piaggio: I think that many pupils who take an interest in manipulation do not like the philosophical side of the subject, but I am surprised to find also that some students who are quite good at the philosophical side are not good at the manipulative. In teaching a class of advanced students, one of my pupils asked me whether it was not the case that the radius of curvature was really infinite, because a curve was made out of little pieces of straight lines. (Laughter.)

Mr. Carson: The point which Dr. Nunn has raised is a large one, and one upon which, as he truly said, we agree in theory; but as we are not now discussing teaching practice, it may well be left to another occasion. Another speaker seemed unduly pessimistic, I thought, as regards the possibility of imparting philosophical ideas to young students. I can only say, from actual experience, that things such as simple general ideas concerning non-Euclidean geometry or the *Schnitt* can be brought home to ordinary boys of 18 or 19 in ordinary schools. Mr. Godfrey, like myself, did not find that he could comprehend such things till after he had taken his degree. And for what reason? Why, for the simple reason (I say this with all respect for his and my teachers, and few men owe more to their early teachers than I to mine) that the subject as taught to us was mishandled. We were victims in the earliest stages of that system of arid formalism of which he has himself been so strong an opponent. What I have tried to express is that, with some definite aims and logical idea of progress in our minds, we ought to recast our system from the start. If we do that, then I can say with some confidence that many of these ideas can be brought home, in some simple form at least, to boys and girls of the age of 18 or 19. If this be attempted on a basis of the old lines of teaching elementary work, or on the present lines, failure will result; and the failure will be more conclusive on the present lines than on the older, because they are more deficient in logic. I wish to put this as forcibly as I can, for to my mind the whole future of our subject depends, in large part, on the amount to which modern views of mathematics can "inform" our teaching in the earliest, and indeed all, stages.

To say that these things are not for ordinary boys and girls, is to deny them to the human race as a whole. They are interesting, fruitful, and most educational for ordinary human beings.

Is it rational so to deny them, in virtue of our own admittedly imperfect experience, without making some effort to make them the common property of educated men and women? May we not, in so doing, commit an error like to that of our predecessors in regard to the calculus, the error of assuming that the limitations of one generation are those of its successors?

The Chairman: I think there is some little ambiguity about the use of the expression "the philosophical side of mathematics," and that the precise treatment of the subject depends very much upon how far one proposes to go. I should be inclined to think myself that many of the ideas can be given or suggested without any rigorous treatment of the whole scope of the subject. I will illustrate what I mean. If you take the case of the irrational number, I think it would be very valuable if a good many of the students in higher classes should, in some such way as that suggested by Mr. Carson, be brought to a clear notion of the irrational number, such as that $\sqrt{2}$ represents a definite entity; but I do not think it is either possible or desirable to attempt to give them the logic of the whole scheme of these entities.

Mr. Carson (interposing): Quite so.

The Chairman (continuing): I should advocate that pupils be brought up to the idea, but not to the treatment of it on a purely logical basis, because, in the first place, it is rather prolix and it would take a great deal more time and attention than they could give to it. Besides, at their age, pupils would hardly appreciate the meaning of it. I tried to indicate the same point in my address in the matter of the notion of a limit. I think the idea of a limit could be brought to the knowledge of boys in a general course, and that it would be really a valuable possession to them, but I do not want to say for a moment that the whole doctrine of limits should be built up on a logical basis, proving all the theorems rigorously. That would be far beyond what could be done in any general course in schools, and I do not think it is necessary. The important point is to impart an idea of a limit, a real idea, not merely of what the word means, but as a distinct concept, without a full treatment of its whole philosophy. I should rather gather that Mr. Carson more or less agrees with that view.

Mr. Carson: Entirely.

The Chairman: I am so far with Mr. Carson that I am inclined to think this aspect of mathematics has been unduly pushed backwards by the extreme reaction we have had against the purely abstract teaching of older days. There is a danger of the pendulum swinging too far the other way. I think a good deal of time would be used with more advantage in a real effort to understand what $\sqrt{2}$ means than in an undue amount of the kind of practical work which is done nowadays. (Applause.)

AFTERNOON MEETING.

PROFESSOR E. W. HOBSON, D.Sc., F.R.S., in the chair.

A PLEA FOR THE EARLIER INTRODUCTION OF THE CALCULUS.

BY MR. C. V. DURELL.

ALL who are interested in educational matters cannot fail to be conscious of the widespread spirit of unrest that is abroad. The energy of this association and other similar bodies is proof enough of the increased interest in the efforts that are being made to examine and recast, in the light of modern work, the existing forms of secondary education, and to grapple with the new problems created by a changing system of external forces. Criticism within this sphere is as healthy as it is inevitable, just because it secures a careful scrutiny of the grounds for proposed alterations; and it should serve as a safeguard against fantastic and ill-considered developments of methods and ideas that under proper limitations may be both serviceable and stimulating. Any proposal to introduce a new variety of an old subject, and still more an apparently fresh theme, into the curriculum can only lead to beneficial results if a thorough investigation of the principles involved and the purpose in view has first led to a discrimination between the essential and non-essential, the useful and undesirable, elements of the subject.

The danger of misinterpretation and its consequent evils is considerable, as is proved over and over again in the history of education. To illustrate my point, I will refer to a single example. One of the principal changes effected in the teaching of algebra of recent years is the introduction of graphical work. The value of this subject lies in the opportunity it affords the student of acquiring in a simple manner the idea of functionality. He is led to consider the continuous variation in value of an expression involving x, as x is made to vary through a given range. And he learns that the graph he obtains is a pictorial representation of an algebraic law. He also comes into contact with other forms of variation which are incapable of precise expression in analytical terms, and easily realises that the most vivid method of illustrating the mode of variation lies in the appeal to the eye which a curve makes. As has been said, we do not want a boy to think merely of a set of points on the paper and a nicely drawn curve—a pattern just sitting quietly on the paper with no life in it; we want him to think dynamically rather than statically: the main point throughout is to insist not only on the statical but the dynamical aspect of quantitative relationships. These words of

Mr. Godfrey seem to me to summarise admirably the spirit in which graphical algebra should be taken. To make it an introduction to the formal analytical treatment of the straight line and circle, or still more to use it for solving arithmetical and algebraical problems, is a development which has lost sight of the real and original justification for the introduction of graphical work into the elementary algebra course.

It is obvious that in a subject so extensive as the calculus, the danger of attempting to cover too wide a field and of including applications which, however attractive in themselves, are either irrelevant to the main scheme or too difficult or of minor educational value, is all the greater. And I have dwelt in some detail on the above illustration because I think that the same principle, which obtains there, is equally trustworthy in the case under consideration. It will therefore be necessary to return to this point later.

A suggestion to modify our present curriculum necessarily implies some dissatisfaction with existing conditions. And it is important to recognise the fact that criticisms are levelled against it not only by many in England whose business it is to teach it, and by many more engaged in educational work abroad, but also by those who have taken it as a part of their own general education. It is also well to bear in mind that English schools occupy a more favourable position than similar institutions on the Continent, as judged by the number of hours devoted to the subject. It is not far from the truth to say that at the present time mathematics is a protected industry in this country. The amount of time allotted to it is due far more to the fact that the chief examining bodies require quite a high standard in elementary manipulative work than to the belief that the subject, as at present taught, is of unimpeachable educational value. Such a position is as hazardous as it is unsatisfactory. A time may well come when the Universities cease to make mathematics, just as much as Greek, a compulsory subject. And under such circumstances, unless there existed a substantial public confidence in its educational advantages, it would take a far lower status in secondary schools, and the time given to it would certainly be seriously curtailed. I find it hard to believe that the contention that the mental discipline it affords renders it indispensable as a school subject is in itself sufficiently overwhelming to convince at any rate the general public. There are many other subjects which can be made to serve the same purpose, although possibly not to the same degree and in the same respect. It is desirable to adopt a wider field of view and base its claims on the cultivation of the imaginative faculties and on the mental stimulus which a richer group of ideas will secure. If you ask a man of ordinary intellectual ability, but of no special mathematical talent,

what he thinks of the mathematical work he did when at school, it is highly improbable that you will hear much in its favour, simply because he never attained the point at which he could realise for himself the purpose of the work he was doing. I am far from wishing to assert that a system is bad under which the ordinary student is unable to make use of what he learns in his practical affairs. No one, save the specialist, will ever have occasion to do so. But I would urge that it is a failure if he is unable to perceive what uses can be made of it, and if his vision of the subject is limited to a manipulation of, to him, unintelligible symbols, and is so contracted as to conceal from him the fruitful and inspiring ideas to which the elementary work leads.

I do not see how any boy can be expected to work keenly and interest himself in what he is doing when he is given no clue to the purpose of the work in hand; and no doubt this is a truism. Every teacher knows that in some measure he must always take his division into his confidence. But unless the ground that is covered is reasonably extensive, it is not easy for the master to do this. The introduction of numerical trigonometry with the allied practical problems on surveying and some course on mechanics, experimental and possibly theoretical as well, into the course of the average boy, has been invaluable to this end. But it still remains true to say that he has to tackle a large amount of manipulative work which he will never have occasion to apply. Such work kills his interest in the subject, and so becomes a deadening instead of a quickening influence upon his mind. It is as valueless to fetter the ordinary boy with this burden as to teach a student who has no ear and little voice to sing. It is equally tedious and painful to pupil and teacher alike. If they go through with it, no doubt it is a good mental discipline, but the benefit ends there.

Every subject contains various ideas which stimulate and interest the student, and for their appreciation some manipulative skill is essential. But by a proper arrangement of the course, this may be acquired in such a way that the student is never far away from the ultimate purpose of the work. Algebra, for example, contains a number of fruitful themes such as problem work, graphical ideas, variation. But the subject does not end there: and the proposition which I wish to lay before you this afternoon is that this section is only properly rounded off and completed if we add to it a summary of the principles of the calculus. A text-book on algebra which omits any reference to the calculus is as incomplete as a treatise on elementary geometry which passes over three-dimensional work. In so doing, a great opportunity for developing the powers of thought and visualisation is lost. But up to the present it has not been fashionable to do so; and indeed twenty years ago it would have been quite

impracticable. But so much has been done of late towards the discovery of easier routes to peaks that were considered within the powers of none but the practised climber, that now it has become clear that it is accessible to all save the intellectually lame and blind. There is nothing strange in this; it is one of the most prominent features of mathematical history. To quote once more from Mr. Godfrey's paper—If the infinitesimal calculus was the high water-mark for the seventeenth century, so was Euclid's geometry for the third century B.C., the Arabic notation in Europe for the twelfth century A.D., and the method of long division for the fifteenth century. Each generation in its turn is able to enrich the ordinary school curriculum by absorbing and digesting something from the province of the specialist, and so increases the efficiency of the educational system; and in the opinion of many the contribution of the present age should take the form of an elementary and feasible scheme of calculus methods.

Much has already been done to solve this problem, particularly abroad. In Austria the curriculum embraces the idea of functionality, including the measurement of the variation of a function by the differential coefficient, and in France the course includes maxima and minima problems and geometrical applications to areas and volumes. But in England the movement is far from general; only a few schools have given it a serious trial. The most notable instance is of course the Naval College at Dartmouth. The subject is there taught to boys of sixteen years of age with conspicuous success. Any one wishing to study in detail the lines on which it is presented should read Mr. Mercer's *Calculus for Beginners*. Those who have already examined it will see that the scheme of work I am about to suggest follows very closely the lines he has laid down, with a somewhat less extensive range. The ordinary text-books, on which probably all present here were brought up, opened with a chapter on limits, which discussed the evaluation of expressions, many of quite a complicated and artificial character, including logarithmic and exponential functions. The subject was in fact introduced as though it had nothing beyond a purely analytical significance, and in such a way as to deny access to all who had not acquired quite a high degree of manipulative skill and a fairly extensive knowledge of algebra and trigonometry. Could anything be more uninviting or more calculated to damp the enthusiasm of the student? Certainly, in my own case, at least one term elapsed, and probably more, before I had the vaguest idea of what it was all about. Probably there are many who learnt, as I did, to differentiate complicated functions, once, twice, and even n times, long before they had any conception of the object of their work or were able to interpret the processes they employed. No doubt

for the specialist, whatever the avenue of approach may be, it will all come right in the end. And he must acquire at some stage or another considerable manual dexterity. But, this being so, it will serve his purpose equally well to acquire it later; and by this plan it becomes possible to keep together the professional and the amateur mathematician, and so postpone as long as possible a bifurcation which ultimately becomes inevitable.

For the sake of the amateur it is essential that from the outset the analysis should be as simple as possible, and of a suggestive character. The idea of a limit is of course the foundation stone; and the first task of the teacher is to illustrate and develop this conception. To this end, it is probably best to resort to a kinematical example. Take the case of a train in motion, its distances "s" feet from a fixed point on the line being noted for given times "t" seconds. If the train travels s feet in t seconds, we may say that it has an average speed of $\frac{s}{t}$ feet per second. Now let us examine the value of this fraction for $t=1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, etc.; if in each case the value is the same, the motion appears to be uniform. But otherwise it is evident that the velocity is varying, and these average speeds we calculate give us only an approximate idea of the rate at which the train passed the fixed point on the line; and the smaller the interval of time we take, the more closely does the average represent the ideal rate or the limiting velocity of the train at the point under consideration. Let us take for instance some simple formula such as $s=\frac{1}{8}t^2$, and enquire at what rate the train is moving after 100 seconds. We should make a table of values for s corresponding to $t=100$, 110, 105, ... and then calculate the average speeds for the intervals of time between $t=100$, $t=110$; $t=100$, $t=105$; etc. We then find that these approximate more and more closely to 40 feet per second, although no interval of time, however small, gives precisely 40 as the average speed. But by taking a sufficiently small interval we can make the average speed approach as near to 40 as we please. We therefore say that the limiting value of the average velocity after 100 seconds is 40 feet per second and call this the velocity, an ideal velocity of course, at that instant. To make this clearer, it would be well to end by calculating the average speed for the interval $t=2$, $t=2+h$, which gives $40+\frac{1}{8}h$ feet per second, and which evidently tends to 40 as h tends to zero.

It is well worth spending a large amount of time on this illustration, for it contains the root-ideas of the subject. Graphical methods may be also usefully employed. After drawing on squared paper the graph of $s=\frac{1}{8}t^2$, with a choice of units that prevents the curve from becoming too contracted, it is

possible to read off from the figure the average speeds corresponding to given intervals. Thus the average speed over the interval represented by the arc PQ of the curve is obtained by producing the chord PQ and reading off the increase in s for a unit increase in t; and the closer Q is to P, the better is the approximation towards the ideal velocity at P, the limiting form being given by the tangent at P.

The function which expresses s in terms of t affords the simplest opportunity of illustrating the rate of growth of a function, for the idea of a body in motion is so familiar. It is, however, well to proceed to other functional forms. Suppose the number of cubic inches of wine in a glass is $\frac{1}{2}x^3$ when the depth is x inches, we may then consider the rate at which the contents are increasing per unit increase of depth. As the depth varies from 1 to $1+h$, the increase in the contents is $\frac{1}{2}(3h+3h^2+h^3)$, giving an average rate of increase of $\frac{1}{2}(3+3h+h^2)$ cubic inches, which tends to the limit $\frac{3}{2}$ cu. inches per inch increase of depth when the depth is one inch. There are numerous types of examples which may be used to illustrate the meaning of this process—the expansion of a circular plate or a spherical soap-bubble, the variation in pressure of a gas, the acceleration of a moving body per unit distance, the deduction of an expression for the area of the surface of a sphere from its volume, the cartesian equation of a curve from its area. These stimulate the imagination of the student from the very outset; all his attention is concentrated on the principles and purpose of the subject, and he is not distracted by analytical difficulties.

So far nothing has been said about the distinctive notation of the calculus. After the work which has just been mentioned, the expression of δy in terms of δx is made very easily. Some informal discussion in class will certainly be necessary in order to make clear in what sense the term "small quantity" is used. If you ask a novice what he means by a small quantity it is highly improbable that he will, unaided, suggest the idea of comparison, but a few examples soon set that right. Professor Whitehead has a few amusing remarks on the subject in his book on mathematics in the Home and University series. To introduce the δy notation we may return to the kinematical example. Taking some simple relation, such as $s=5t+3t^2$, we calculate the distance corresponding to the time $t+\delta t$ and call it $s+\delta s$, which gives an opportunity for explaining the use of the functional symbol δ. The average speed $\delta s \div \delta t$ is then found to be $5+6t+3\delta t$, showing that the limiting value of the velocity after t seconds is $5+6t$ units. The limit of the ratio $\delta s \div \delta t$ when $t \to 0$ is then defined as $\frac{ds}{dt}$. It is obviously essential to spend some time in explaining the meaning of this symbol: it is

useful to point out that the ratio $\delta s : \delta t$ may never be actually equal to $\frac{ds}{dt}$, as in the example under consideration, and that the symbol $\frac{d}{dt}$ is just as much a functional symbol as δ or $\surd$ or log, and is introduced merely in order to have a compact method of writing down the limit, when $t \to 0$, of the ratio of the increase in s to the increase in t.

Another set of numerical illustrations should again be used at this stage. These should include the idea of the gradient of a curve and the symbol $\frac{dy}{dx}$. It is also worth while considering the special case of the gradient of the linear equation, analytically and geometrically. The student is then in a position to write down the equations of the tangents to curves of the form $y = a + bx + cx^2 + dx^3$, the examples being of course numerical and moderately simple.

It is almost superfluous to say that there is a real danger of overloading the student with numerical work. If he is required to do a great deal of computation in the early stages, it occupies a large amount of his time, does not really assist in clearing away difficulties, and certainly introduces a risk of boredom. Some he must of course do, but much can be supplied by the teacher.

Every question has up to this point been solved by a recourse to first principles. The pupil should by now have been sufficiently grounded in the ideas of the subject by the previous work to be able to appreciate the significance and value the time-saving qualities of a few general results. He should first proceed to the differentiation of x^n for the values $n = 4, 3, 2, 1, 0, -1, -2$, and possibly he might be able to take the more general case where n is any positive or negative integer, as it is easy to find proofs which do not require the Binomial Theorem. But for practical purposes the restriction to a few numerical values of n will not be found to curtail the work at all seriously. Certainly any cases, other than these, may well be left out of consideration, the success of the work is quite independent of them. There are two other theorems which must be taken: the differential of the sum of two functions and also of the product of a function and a constant. The differentiation of a product or quotient of two functions is quite unnecessary. This machinery may now be used to work rapidly through examples that have previously been solved from first principles, thus forming a revision course. This also provides a convenient opportunity for illustrating the result that $\frac{dv}{dx}$ is equal to the ratio of

$\frac{dv}{dt} : \frac{dx}{dt}$. Suppose we consider a reservoir whose volume v cubic feet is a given function of the depth x feet, and are required to determine the rate at which the surface is rising when a given number of cubic feet of water enter per second. Here $\frac{dv}{dt}$ and $\frac{dv}{dx}$ are given, and the problem is to find $\frac{dx}{dt}$. By employing the δ notation all difficulty vanishes, and the student is led to realise something of that very mysterious formula $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$.

The second stage of the work introduces the student to maxima and minima problems. The change in sign of $\frac{dy}{dx}$ is best illustrated graphically, but in addition to this, little difficulty is experienced in understanding the method depending on the use of the approximate equation $\delta y = \frac{dy}{dx} \cdot \delta x$: and it is important to make clear the fact that the sign of $\frac{dy}{dx}$ is a criterion of whether y is an increasing or decreasing function of x. The test involving $\frac{d^2y}{dx^2}$ provides a useful exercise on the same idea. As examples we may take in addition to the ordinary maxima problems some simple curve tracing, and the calculation of errors in results due to small errors in observation.

The last stage deals with the inverse problem; given $\frac{dy}{dx}$, what is y? Here again, to explain the introduction of the constants, we may return to kinematics. The student has already learnt the meanings of $\frac{ds}{dt}$ and $\frac{d^2s}{dt^2}$; take the equation $\frac{d^2s}{dt^2} = g$, and make him integrate it. He then sees without trouble that the constants enter owing to the absence of any data relating to the initial conditions, and are determined as soon as these are known. Other similar examples may be taken, and such problems as given the slope of a curve, or the subnormal, etc., find its equation, are equally interesting and instructive.

We next proceed to areas. A figure readily suggests that $\delta A = y \,.\, \delta x$ approximately, or $\frac{dA}{dx} = y$, and at this point we may introduce the notation $A = \int y\, dx$. This work should be illustrated by a number of special cases; the simplest is, of course, the straight line through the origin. And naturally this leads to the summation idea, showing that $\int_a^b y\, dx$ is equivalent to the

limit of $\sum_a^b y\,\delta x$. As far as possible it is preferable to work with definite rather than indefinite integrals, and the geometrical illustrations make it easy to do so. With the work on areas, it is natural to associate volumes; what further illustrations are included must depend on the time at the teacher's disposal. But it would be a pity if it were necessary to omit simple examples on centre of gravity and work.

This concludes the course that I believe is well within the compass of the ordinary boy. Perhaps there will be some who will consider that it is unduly abbreviated. It contains no mention of the trigonometric or exponential functions; even the differential of such an ordinary function as $\sqrt{x^2+a^2}$ is excluded. And it might be urged that an additional half-hour would be enough to embrace such cases as these. Perhaps it might; but a student who could perform this operation would have no firmer a grasp of the principles of the subject than his more ignorant brother. As Mr. Mercer has said—"I had much rather a boy, confronted with a problem, should, after analysing it, be able to say, 'If I could differentiate (or integrate) this function of x, I could solve the problem,' than that he should be able to perform the operation without seeing its bearing on the question." Such a course as that outlined above has the advantage of employing very simple machinery, and yet opens the eyes of the student to the wonderful and powerful methods of the calculus. From personal experience, I am fully convinced of the feasibility and the beneficial effects of the work. It is a subject which attracts and interests in a remarkable way the boy of no special talent who has exhibited merely a polite boredom when faced with discount, stocks, or elaborate fractions and indices or homogeneous equations of the second degree; and the time required for the complete course is about two hours a week for two half-terms. Before sitting down, I ought to make some apology for the length of this paper, and for what may appear an excess of detail in the earlier portion of it. This is due to my fear lest it should be supposed that I were advocating anything but the simplest and most elementary form of treatment. This is essential to the success of the proposal, and in my opinion ensures it.

THE DISCUSSION.

Mr. A. W. Siddons (in opening the discussion) said: I can imagine how Mr. Durell's words would have been welcomed by some of our colleagues. I remember recently hearing of a mathematical master who announced that the calculus was going to be taught at about the middle of a great public school.

"What! the calculus?" The master who spoke was a classical master who regarded the calculus as the coping-stone of the whole of mathematics. Well, he naturally would not press his point, but would leave it to be settled by the mathematicians, though he would, no doubt, think they had all gone mad. Then there is your mathematical colleague who, on being asked to teach calculus, replies, "Why, I have not done any since I came down." What will he do? If he looks up his Todhunter and feeds his boys on that they are lost. But tell him that what he is to teach is just what has stuck in his own brain! I once learned chemistry, and knew something about it, but I do not remember anything of it now. Still, it made an impression on me. Something has stuck; and in the same way your mathematical colleague of whom I am speaking (I do not refer to any particular one)--(laughter)—would have been left with the impression of the calculus. It would have left him with a certain sense of general principles. That is just what we want taught. I am not going to consider details of the calculus, but rather how it is possible to fit in the teaching of it under existing conditions. It is part of the duty of this Association to put forward ideals, and also to see how much is possible at the moment. I will deal with the question, How is it possible to teach the calculus with our existing examination system? The obvious reply is that the examinations must be thrown overboard. That is impossible at present; but I think it is our business to try to throw over the examination systems where they stop the development of our work. (Hear, hear.) We hope for changes. But how is it possible to fit calculus in at present? The solution I would suggest for consideration is that we should avoid teaching many of the things that are not essential, and that our bright work should be put much earlier in the course. For instance, we are now teaching practical mechanics to boys in the middle school at Harrow, whereas a dozen years ago only the top divisions took mechanics. Much may be done, I think, if work of that kind is taken earlier, and if we forget examinations for a time.

How can it be done? I think one of the first things to do, at almost every school where I have seen work, is to increase the pace and accuracy of calculation. I am sure a great deal would have been done if, in the earlier stages, boys and girls were taught to be much more rapid in mere computation. Five minutes' practice of this kind every day for two or three weeks has an enormous influence both on pace and accuracy, and would save enough half-hours later on to enable us to introduce the whole of Mr. Durell's course. Apart from that, we can cut down the syllabus very considerably. We can avoid a great many technicalities. Personally I should like to cut out of middle school work all Stocks and Shares, Present Worth, and things

of that kind. If they are regarded as necessary, come back to them afterwards; certainly boys are riper for such work at sixteen or seventeen than they are before that age. It is the same with algebra and trigonometry. We are compelled by various examinations to teach a great deal that some of us think is undesirable. In the middle school let us take the parts of the subject that develop a boy's mind, and do not necessarily make him skilful. All the technique can come at a later stage when the examination is nearer, and when a boy, being older, can acquire skill more rapidly. You save time by putting such work later. Geometry has been very much reduced already. I suppose to-day most of us include in the geometry course only as many propositions as there are in Euclid's first book alone. This reduction of time necessary for geometry is very considerable. I think the practice of doing too much drawing is gradually adjusting itself.

With this saving of time it would be quite easy to include in the middle school curriculum the calculus course sketched out by Mr. Durell, and a certain amount of statics and dynamics. But you do not want to drift into too great difficulties here. I would limit the statics to experimental work and easy numerical applications of the law of moments, the triangle or parallelogram of forces, centres of gravity, machines, and the ideas of work and friction. In dynamics the ideas of linear motion would be studied in connection with the calculus; projectiles should be included, and I should like to introduce some ideas on force and work and energy.

What I have been aiming at is to suggest that we should consider a course which is possible under present examination conditions. If the kind of work I have indicated is done in the middle parts of schools, then afterwards, if needs be, we could consider the examinations; but I think the examinations can be put off so far as the middle part of a school is concerned. I think our work in school is too much dominated by the fear of examinations. Let us frankly call it cramming. Cramming for examinations is much more successful if it is not begun too soon. Try to forget examinations as long as possible.

Finally, may I say that I have taught calculus somewhat on the lines which Mr. Durell has suggested for close on ten years? I have taught it to boys who have afterwards got scholarships at the universities. I have taught it—I won't say to the dullest of the dull—but I have taught it to boys of all sorts of ability, and it is astonishing how keen they are and what a grip they get of the application of the ideas underlying it. Professor Hobson spoke of the value of the idea. We want to get the idea without the technique. As Professor Hobson has said, it is not desirable that we should build up a complete, logical, rigorous

theory of limits, but rather that we should give the idea of what a limit means. I feel that a great deal of such work could be done even under existing conditions if people were not so afraid of examinations which are to come two or three years afterwards. Let the examinations be considered only when the boys get very much nearer to them.

Professor Hudson: I think, sir, we have had an example to-day in the two speeches we have just heard of how the wheels of time go round. Mr. Durell has been describing the very method by which his father learned the differential calculus. The method of teaching appears to have changed in the meanwhile, and now we are coming back to it, and Mr. Siddons has described, I think, the very attitude which was adopted by teachers in schools of the date to which I have alluded.

Rev. E. M. Radford: It is my lot in life to teach the calculus to training-college students who are working for definite examinations, and therefore we cannot limit our range in quite such an elementary way as Mr. Durell described. When I thought of taking part in the discussion, I certainly did not know that we were going to use the word "calculus" in that very limited sense. Of course there is always a certain amount of ambiguity about the use of the term, but it does seem to me that if you are going to teach the beginning of the calculus to older pupils, you must assume a greater amount of knowledge on their part, and you must propose to attack it in rather a different way than you would if you had the chance of beginning it quite early with schoolboys. And so the question is as to how soon the calculus should be taken in such cases. Are we to begin as soon as our students are familiar with the elements of algebra and very early trigonometry, or are we to wait until they have at least some conception, however limited, of the higher parts of these subjects? That is to say, are we to teach only the algebra of the finite before we start the calculus? We have heard a good deal to-day about limits and the fact that the differential coefficient involves the idea of a limit, which is itself inseparably allied to the conception of an infinite series. It is very difficult to give an older person a simple idea of what a limit is; that is to say, to persons who have been taught algebra in the way that most of my students have been taught—in quite a wrong way. I mean before they come to me. (Laughter.) I have not got the public-school type of boy to deal with now. We can see at once that functions which can be differentiated without any kind of knowledge of an infinite series can be counted on the fingers of one hand. We cannot be content with the general ideas as apart from the rather more detailed application. Anybody who has grasped the idea of the differential coefficient will not be content with it as a mere

abstraction; nor will he be content with being able to differentiate such functions as "x^3" or even "$\sin x$." Something more will be wanted. It is like putting into a person's hands a very delicate piece of machinery, explaining its use, and telling him he is on no account to use it, or that there is no material to which that machine can be applied. Therefore we come back to the question as to what is to be assumed and what is to be regarded as known, and I hope the Association, at some time or another, will come to some common platform in this matter. This leads me to consider what I believe is the great stumbling-block to any very early introduction of the calculus, viz. the exponential function. There is no question on which the text-books are more at variance, and I believe the same may be said about mathematical teachers. It is a question which cannot be ignored or slurred over. Any intelligent boy will at once ask whether there is any function which is its own differential coefficient. A teacher told me the other day that when he was doing elementary logarithms he took the expansion of e^x *incidentally!* He did not explain how he did it.

In fact, the difficult question is—how is the symbol e, which is at the bottom of the work, to be defined? Can such a quantity be left or defined merely as an approximation? I do not think it can. The whole difficulty lies in proving that $\operatorname{Lim}_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ is finite and definite. You cannot get over the difficulty by defining $\log x$ as an integral. Even then you get back to the same thing in the end. Now, I know the way it is usually approached by many teachers, for whom I have the greatest admiration, is by means of numerical calculations, that is, by calculating approximately the values of $(1+\frac{1}{10})^{10}$, $(1+\frac{1}{100})^{100}$ and so forth, by means of Logarithmic Tables, and then concluding, from this evidence, that the expression does tend to a limit in a definite way. This method always reminds me of the Oxford man, who had been in for Responsions. After the Euclid paper, a friend asked him how many propositions he had proved. "Well, I should not care to say that I proved any, although I made several of them appear extremely probable." (Laughter.) That is all you do in dealing with a limit in such cases. You make it appear probable. I do not think intelligent students will be willing to build the whole of their subsequent work in the calculus on a basis which is so hazily laid out. We must be on firmer ground. But, granting all this, we are not out of the wood. I came across a quotation a day or two ago from Mr. Hardy of Trinity, in which he says that the theorem

$$\operatorname{Lim}_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x,$$

with whatever degree of rigour it may be stated, is essentially accessory and not fundamental, and to base the exponential theorem upon it, although possible, is logically quite wrong. Upon what then *are* we to base it, before introducing the exponential function in the calculus?

I merely deal with this one point to indicate to teachers one of the many uncertainties involved in this question of the early introduction of the calculus. For the sake of the future credit of mathematical teaching, do not let us try to teach our pupils to run before they can walk. That seems to me to be the danger. And this discussion cannot fail to bring home the necessity for some corporate action on the part of the Association in this all-important matter, which will carry weight with those examining bodies that we all should like to, but cannot, get rid of, and of course with the Universities as well. (Laughter.)

We know what the Association has been able to accomplish already in the teaching of arithmetic and algebra and geometry. Every member of the Association is quite convinced that the Association has justified its existence over and over again in this direction alone. This teaching of the calculus will be the next great question that presses upon us, and I venture to suggest that it shall be dealt with in the same way as the others have been dealt with, resulting in as much success. I mean by a committee on the teaching of the calculus, to be appointed by the Association, and that this committee shall draw up a series of recommendations for the consideration of the Association, so that we may be able to take a common line of action. (Applause.)

Mr. Dobbs: Mr. Radford suggested that the chief difficulty is the defining of e. One method of overcoming this difficulty is as follows:

I. It is easy to prove that if OE is drawn from the origin O to touch the curve $[y=a^x]$ at E, then the ordinate of E is independent of a, *i.e.* the ratio FE/OA is an absolute constant. Define e as this constant ratio.

II. There is no difficulty in showing that the subtangent in the same curve is a function of a only, namely $1/f(a)$, where

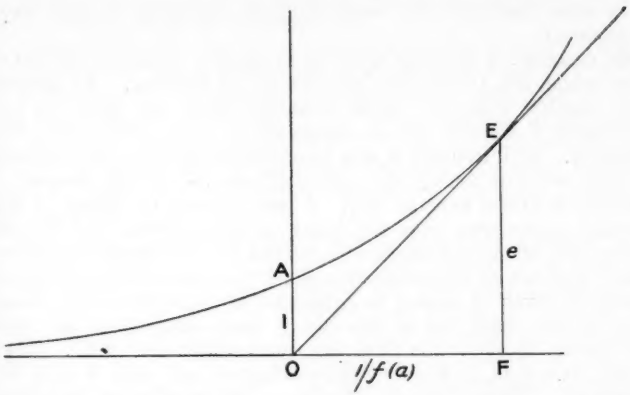
$$f(a)=\operatorname*{Lt}_{h\to 0}\left(\frac{a^h-1}{h}\right).$$

III. Now take e as the base of logarithms, and show that

$$f(a)=\frac{1}{OF}=\log a,$$

and hence that $de^x/dx=e^x$.

(See *Mathematical Gazette*, Jan. 1910, vol. v. p. 179.)



Mr. A. G. Pickford: I should like to raise a question on a point which was not clear to me. I did not understand whether Mr. Durell, in the course of his paper, used the language of notation for the sake of brevity in speaking to mathematicians, or whether he was proposing that so much notation should actually be included in elementary calculus for quite young boys. I have been in the habit of teaching a great deal of differential calculus, as far as possible without notation, to boys of an earlier age than that at which they are supposed to learn it in the ordinary way, and I am strongly convinced of the great importance of familiarising boys with the ideas of the calculus before they are brought to study any formal course in the subject. For instance, when we have drawn a graph we apply the ruler and find the maximum point on the graph. Then I say, "We will calculate the actual position of this maximum point, and see whether it agrees with our practical observation. If the class as a whole cannot follow, the better boys will please pay attention." To mention "better boys" has a wonderfully stimulating effect. (Laughter.) I am convinced that during at least a year of a boy's school life, ideas of the calculus should be introduced in such a way, and without formal study.

One of the suggestions I should like to make as a practical teacher is, that in the teaching of mathematics we should not be afraid of letting ourselves go. I am not ashamed of the fact that very often I spend as much as half a lesson in doing something which really the boys ought not to be doing in regular course. Some problem, of calculus it may be, is suggested by a problem in the set lesson, and by continuing my work on the board I follow up the line of thought, and let the boys understand as much of it as they can. I am

quite sure that one does very valuable teaching in that way. (Applause.)

Mr. Godfrey: I am very much in sympathy with the outline of work which Mr. Durell has laid before the meeting. As regards the e trouble, I do not know what his views are about it, but personally I should not be ashamed to run away from the difficulty. If I thought it was unprofitable to teach the calculus to boys without facing the difficulty of e, I should despair of teaching it to the average boy. I take it that the point of Mr. Durell's paper was, that the ideas of the calculus can be put before boys with good effect without the use of difficult functions. Really it is not an inconsistent point of view to avoid complicated functions; even in regard to equations, boys learn to solve these to a certain stage, but at the end of their study there are very few classes of equations they can successfully deal with. They do not, as a rule, go higher than quadratics, and that is only the beginning. They cannot look at a transcendental equation. Personally I should attempt to teach calculus without e.

There is just one other point. Mr. Radford seemed to suggest the formation of a committee to report on the whole subject. I very much doubt if the time is ripe for that. Just now we are very much feeling our way. I am afraid that if we form a new committee, there is not yet enough basis of experience on which to proceed, in order to issue a report which would carry authority. I am very much in sympathy with this movement, but I should rather fear that to tackle this subject prematurely would be to our disadvantage.

Prof. E. V. Huntington (of Harvard) spoke briefly of the value of such discussions in bringing about cooperation between School and University teachers, and in securing agreement in such questions as that of suitable notations in the calculus.

Mr. Stephenson: From my own experience I should like to speak on one feature of the advice which Mr. Durell has given us. He has spoken about the use of dynamics for purposes of illustration. My own experience of boys—such as it has been—has led me to the conclusion that to endeavour to teach the meaning of $\frac{dy}{dx}$ from dynamics is likely to lead to confusion. I have not found that boys take readily to the conception of velocity. A boy is singularly unwilling to use his imagination to conceive the expression, at all events unless he is in a school where he is taught practically in the laboratory. I think that boys, instead of considering the velocity of a train at a given moment, generally prefer to consider the number of miles an hour at which it is travelling. At any rate, my own experience teaches me that the velocity of an object which is continually changing is one of the most difficult conceptions to enforce so

far as the ordinary boy is concerned, but with most of Mr. Durell's paper I am completely in sympathy.

Although I agree that most examining bodies are rather unsafe guides, I think there is one body of examiners who have chosen their curriculum remarkably well and with a great deal of foresight—I mean the examiners for the Army Entrance Examination. In the examination for admission to Woolwich a certain amount of calculus is set, and it has been my duty during the last few years to teach, I am glad to say with a certain amount of success, the calculus to boys.

As regards the '*e*' difficulty which has been referred to, I have been much struck by the (to me) entirely new theorem, a very interesting theorem, which has just been explained on the board. I do not feel, however, that I shall be able to use it in my own teaching. I do not think it would be altogether a suitable way of introducing the subject of *e*, if it is to be introduced to boys at all. The thing seems to me to be too artificial. Perhaps I may say a word or two about the way in which I have dealt with it myself. I define *e* as 2·7183. Subsequently I explain *e* as the sum of the series $1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\ldots$. At a subsequent stage I give my pupils the theorem that *e* raised to the power of x is equal to the series. As regards the corporate action which the Association has been asked to take with regard to bringing forward a scheme for teaching the calculus, I think we must recognise that we are face to face with the following question. At any rate, we must bring ourselves face to face with it. Is it, or is it not, an immoral thing to teach the result of a thing independently of the proof? We *can* teach its applications a long time, possibly a very long time, before the method of proving the theorem is taught. I think that is a question which ought to be faced. If some attempt were made to answer it, it would be helpful. Personally I have no hesitation in saying that I do not consider there is any difficulty about teaching a theorem and its applications, and explaining it, entirely independently of giving any proof. I may add that the use of tables of logarithms is an example familiar to us in which a very large number of theorems are made use of as a matter of course, although they have never been proved by the pupils who use them. Of course, subsequently in one's teaching, one would go on, if there was time, to give the ordinary proof for the series *e* to the power of x. All that would come later.

It is necessary in explaining how to differentiate x^n to use the binomial theorem. I do not for a moment see how, without using the binomial theorem, you are going to prove the formula for the differentiation of x^n. I used to state the binomial theorem with-

out proof and illustrate it by examples. When the students are familiar with the series for $(x+a)^n$, it is possible to go on and prove the ordinary formula. Of course the thing could be illustrated by much measurement and graphical work.

Another point with regard to the sine of x and the cosine of x. These are things which lend themselves to interesting treatment by ordinary geometry. They do not require the use of a series at all. Afterwards, having given these proofs, I used also to give the infinite series and differentiate each of these. I do not think I ever had time to prove these series for the sine of x and the cosine of x. They had to be taken on trust. I do not think there is anything immoral in this if one's mathematical course demands that one takes results on trust.

The Chairman: I think we may take it that Mr. Durell was speaking mainly of a course on the calculus which would be suitable for the ordinary boy in school, and I certainly feel very strongly that, to achieve that purpose, it is quite sufficient that the teaching should be confined to simple elementary functions. I do not think one can expect to arrive at a level at which any very complicated apparatus shall be made available. The point is that, for general education, one must confine oneself to just getting fundamental ideas. Anyone who wants afterwards to fill them out for any special purpose will have to supplement his information. For instance, no doubt there would be higher classes where it would be desirable and necessary to use functions which involved the exponential theorem and logarithms, but I should strongly deprecate the notion that that should be done unless the boy has studied the theory of, and what is meant by the exponential function, and knows the real definition of the number e.

We have heard a good deal of the difficulties about e. I quite agree that they are such that it ought to be banished from the elementary course for the ordinary student, but those who do find it necessary to consider e and the exponential theorem should, I think, study a proper proof of that theorem. I do not say that they should remember the proof of it and be able to write it out. That is quite another matter. I do not think it is in the least necessary that boys should be able to sit down and write out a really good proof of that theorem. That is one of the artificial things which have been imposed upon us by examiners. I do think that at some time or another he should understand the proper proof, and, by seeing the working at every step, understand how that proof is arrived at. He has then done it once for all, because he knows how it is obtained. It does not mean that it should become a burden on his memory, but simply that he has obtained certain knowledge and knows how to make use of it.

In regard to the suggestion that the use of things which a boy cannot prove is immoral, I should not describe such a practice as immoral. I should describe it as uneducational, because I think that to work with a view to understanding the proof would be the very best use boys could make of their time, and because I do not think it would take a great amount of time. I am speaking of the boys in higher classes, who are doing something more than the minimum which Mr. Durell has contemplated. I think they might study the theory of logarithms and the exponential function in the way I have suggested. They could then go on to apply their knowledge with perfect satisfaction.

We have heard a good deal about notation. Professor Huntington spoke about the notation of a differential coefficient. He said there was a great difficulty because $\frac{dy}{dx}$ does not mean the same thing as $\frac{a}{b}$ in algebra. There again it is most important that a boy should understand, when he gets to this stage, that the notation $\frac{a}{b}$ does not mean anything of itself. It has only a conventional meaning. To say that $\frac{a}{b}$ means a divided by b, as if it were a law of nature, is to give an entirely false impression. Professor Huntington seems to think that if a boy has this impression we ought not to disturb it. I think myself that it would be rather desirable to disturb such a belief. In ordinary algebra we assign a certain meaning to a divided by b, but convention has decided otherwise in the other case. I have no strong prejudice in favour of that notation, but I do not think the reasoning is sufficient. I think it is a very good thing to upset fixed ideas of that kind in the boy's mind if he sees a law of nature where he ought to see only a convention which is of an arbitrary character.

We have heard a good many suggestions as to ways of getting round some of the difficulties, and I confess that one hears a little too much of that kind of thing. No doubt there are rather ingenious and interesting ways of getting over the difficulties of the exponential theorem and so forth, but I suggest that it is better to go straight at them, and that the educational value is greater if we do not use these ingenious processes for getting rid of our difficulties. The best educational results will be reached by not being too ready to introduce easy devices. I know that experienced teachers will have views of more value than I can have on such points, but I do not think it is always wise to simplify things by means of such ingenious methods. I think one is apt just to empty out the educational part of the subject.

I do not want it to be understood that I say this in regard to any particular suggestion that has been made here this afternoon, as I am not in a position to form an estimate of the exact values of various methods of teaching, but I do desire to mention these doubts. I think the discussion has been prolonged sufficiently, and, in accordance with our programme, I shall not ask any one to continue it.

I am requested to make a statement in regard to one matter which we have been discussing to-day. The International Congress of Mathematicians will be held at Cambridge in August of this year. I am asked to state the conditions of membership. There are no conditions except the payment of a sovereign. Any one may be a member of the Congress upon payment of that sum, which is the same as one pays for membership of the British Association and similar Congresses. (Applause.)

Dr. T. P. Nunn: While I am most heartily in agreement with the proposal that the school course in mathematics should normally include some instruction in the methods of the calculus, yet I hold very strongly that no attempt should be made to teach the calculus to all pupils in the form of a separate mathematical subject, as (for example) trigonometry and mechanics are taught. What has now become the doctrine of the differential and integral calculus began in the seventeenth century in the form of methods introduced into ordinary algebraic arguments wherever they happened to be useful. The systematic elaboration of these methods into a separate branch of mathematics was a subsequent development. Our true policy is, I hold, to take our pupils back to the point before the "purism" of mathematicians, of which Mr. Godfrey told us this morning, had made a logical cleavage between the methods of algebra and the methods of the calculus. For example, a boy ought to be considered free to use in a mathematical argument the special connection between the functions x^n and $\frac{1}{n+1}x^{n+1}$, just as he uses the special connection between the functions $(x+a)^2$ and $x^2+2ax+a^2$. My recommendation is, in a word, that we should abandon the ordinary notation of the calculus altogether. There can, I think, be little doubt in the minds of those who have taught the subject that this notation, useful as it is in the later stages, is nothing but a source of difficulty and confusion to the beginner. Even if a boy is to specialise later in mathematics, he will probably have clearer ideas about the fundamentals of the calculus if they are presented to him unencumbered by the difficult and misleading notation.

My own practice is—following here the clue of history—to teach integration before differentiation. Any one can be made to understand without difficulty that if the ordinates of a curve

follow some definite law, such as $y=x^2$, the area of the curve from the origin up to a given ordinate follows some other definite law—in this case $A=\frac{1}{3}x^3$. I express this result by calling the former an "ordinate function" and the latter the corresponding "area function," and the two following notations suggest themselves as convenient:

$$A(x^2)=\tfrac{1}{3}x^3 \text{ and } O(x^3)=3x^2.$$

It is an easy transition to the discovery that the relation which holds good here between a moving ordinate and the area generated by it also holds good between a moving area following the law $A=x^2$ and the volume which it generates, and that the converse of this relation holds good between the ordinate of a curve and the tangent at the same point. Thus to the above specimens of an obvious notation we may add the following:

$$V(x^2)=\tfrac{1}{3}x^3, \quad S(x^3)=3x^2,$$

V being read "volume function" and S "slope function." But since the relations between these pairs of functions are identical, the advantage of distinguishing between them soon disappears, so that we finally adopt the symbol D ("derived function") to represent alike the relation between volume and generating area, area and generating ordinate, ordinate and slope. The notation D^2 follows naturally as a symbol for a leap from a volume function to an ordinate function, or from an area function to a slope function; D^3 as a symbol for a leap from a volume function to a slope function. Steps taken in the opposite sense are equally naturally represented by D^{-1}, D^{-2}, etc. No doubt other notations more simple and effective than these could be devised, and I think it would be a pity to adopt a stereotyped plan. My point is merely that it is possible to put the ordinary schoolboy and schoolgirl into real possession of the powers which the ideas of the calculus give without compelling him to master a difficult technical notation.

THE THEORY OF ORDER, AS DEFINED BY BOUNDARIES.

IV. Transformations.

Pursuing the comparison with the ordinary methods of Geometry, we may now say that we have established the foundations of Projective Geometry, and have now to make the transition from projective to metrical methods; the distinction between which, though commonly assumed, has never, so far as I am aware, been clearly expressed. The corresponding distinction in the theory of Order is that between unique collation in general, and what I shall now proceed to define as *transformation*.

In the first place, let me introduce the conception of continuity in respect of collations, or transformations, considered as changes of the names of units. What we have hitherto regarded as a discrete change in the names of all, or all but one or two, of the names of units, we shall in future regard as merely the final stage of a continuous process of change, during which every name "passes in review" from what it was at first to what it is finally, while at every instant during the process the names of the respective units are uniquely collated with those they bore initially. The geometrical analogy of such a process of change would be a kind of flow of the names along the points of a line, the rates of flow of the names past the various points being, however, restricted in ways which it would not be quite easy to define geometrically, at all events without the aid of a plane in which to carry out geometrical constrictions, or the aid of the conception of congruence, which, however, is precisely the conception which we are seeking to elucidate. This analogy shows us, however, that we have not yet got at that conception; for by mere unique collation any part of the catalogue containing three names may be, so to speak, stretched or compressed, so that it may be fitted on to any other three units. The restricted class of collations which correspond to congruent motions, and which I call *Transformations*, may for our present purpose be defined as *unique collations of the conjugate pairs of a given system*. That these do so correspond I proceed to show.

We may, as has been shown, determine a system of conjugate pairs by naming an inversion. Having so established a conjugate relation, so that each unit uniquely determines its conjugate, we may proceed to regard the conjugate pairs themselves as units of thought, and pass in review the *group of conjugate pairs*. During a transformation I shall always assume this to be done in a particular way. We know by theorem 20 that reversion with respect to a conjugate pair does not alter the conjugate relation. And two or any even number of such reversions will leave the resultant collation a positive one.

(22) And since I shall show that: *The resultant of any even number of reversions with respect to the conjugate pairs of a given system may be uniquely determined by naming the collation of any one conjugate pair*, we may assume a transformation which results in collating two given conjugate pairs to have been effected either by simply reverting the first into the second, and then reverting once more with respect to the second to correct the sign, or else by an infinite number of such double reversions, each of which converted the name of the initial given pair of units into the name of a contiguous conjugate pair. It is this latter continuous process of change which I call Transformation.

I may indicate the proof of the above theorem. In the case of a positive system of conjugate pairs, to name any unit is implicitly to name its conjugate, and also the conjugate pair which forms a quartette with them. Now, even while we are regarding conjugate pairs as units of thought, to name a conjugate pair is not quite the same thing as to name an unit. For example, if (aa') and (bb') are two conjugate pairs, we may distinguish collating a with b, and a' with b', from collating a with b', and a' with b; although in both cases the pair (aa') was collated with the pair (bb'). And note, the distinction is not one of sense; we can have no distinction of sense with less than three units, and a conjugate pair has only two. It is convenient to speak of the collations of the conjugate pairs in the above two cases as *the same, but reflected*. So we may say that if we revert a quartette with respect to one of the conjugate pairs, the other remains the same, but is reflected. Reflexion is so far like a change of sense, that two reflexions neutralise

one another. After any even number of reversions, therefore, there will be no reflexions of one of the pairs of a quartette, if not of the other; inasmuch as a quartette, being determined by three units, has sense, which is reversed if we reflect one, and not both, of its conjugate pairs. So long as we know that the number of reversions was even, therefore, a quartette is uniquely determined by one of its units, or by one of its conjugate pairs, considering them as units of thought. To be given the collation of one unit or conjugate pair is, therefore, enough to determine the whole collation.

In the case of a negative system, a single reversion leaves the conjugate boundary the same, but reflected; a second reversion restores it identically. After any transformation, therefore, we have always the identical collations of two units given; if, in addition, we are given the collation of another unit or conjugate pair, we have enough to determine the whole collation.

The remaining case, that of the zero collation, is not, however, quite so simple. At first sight the data would seem to be insufficient, as we are given only two collations, that of one unit besides the identical collation of the self-conjugate unit, which is in both the pairs. It must, however, be remembered that a zero system is a limiting case of either a positive or of a negative system, and what is true of both of them must in general be true of it also. To realise this in the present instance, consider the scheme:

$$\left.\begin{array}{llllllllllllllll}
(1) & & & A & & p & & B & (s) & q & C & r & (D) & w & (s') & & p' & & w' \\
(2) & C & q & B & & p & & A & & & & & & w' & & & p' & q' & w \\
(3) & B & q & C & r & & D & & & & & & & w & & r' & & q' & w' \\
(4) & & & D & r & & C & & s & & & & A & w' & s' & r' & & & w \\
(5) & & & A & & & & & s & & & & D & w & s' & & & & w'
\end{array}\right\} \ldots(g)$$

Here are represented four successive reversions with respect to the conjugate pairs (pp'), (qq'), (rr'), and (ss'), all belonging to the same negative system of conjugate pairs, as is shown by the columns of ws and $w's$, which represent the self-conjugate boundary of the system, being reflected by each reversion. The fourth reversion is represented as bringing A, which has successively been changed to B, C, and D, back to A again. As (1)-(5) collates A, w, and w', all of them identically, it is the identical collation, and we can write in (s), (D), and (s') in line (1). Since (ww') is the self-conjugate boundary of the system of conjugate pairs, the conjugates of all the letters not marked with accents are separated from them by (ww'). Hence if we suppose w' to pass along towards the accented letters and w, in the limit all of them will coalesce together into the single self-conjugate unit of the zero system of conjugate pairs.

To find the actual collation resulting from a transformation of A into X, in a system of zero conjugate pairs whose self-conjugate unit is W, I give the following rule, the proof of which I leave to the reader:

(23) *Let a represent any other unit; let W revert into y with respect to (aX), and let A revert into x with respect to (Wy); then the unique collation required is represented by the scheme—*

$$\left.\begin{array}{llll} (1) & W & A & a \\ (2) & W & X & x \end{array}\right\} \ldots\ldots\ldots\

the transformation which changes (aa') *into* (bb') *will also change* (xx') *into* (yy').

For, consider the following scheme, in which conjugate pairs are treated as units in the scheme, and no notice is taken of possible reflexions—

$$\left.\begin{array}{llllllll} (1) & & & (pp') & aa' & bb' & xx' & yy' \\ (2) & yy' & xx' & bb' & aa' & (pp') & & \\ (3) & (pp') & aa' & & xx' & yy' & & \\ (4) & & & aa' & bb' & & (yy') & \\ (5) & aa' & & & yy' & & & \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

Line (2) is a reversion of (1) with respect to (aa'); lines (3), (4), and (5) are reversions of (2) in which (aa') reverts into (xx'), (bb'), and (yy') respectively. It will be noted that since we are dealing with conjugate pairs, not single units, these collations are completely determined by the data named. And, the collations of lines (3), (4), and (5) with line (1), having been effected by double reversions with respect to conjugate pairs, may be regarded as the results of transformations of (1). And since we are given that the transformation which changes (aa') into (xx') also changes (bb') into (yy'), we write (yy'), in line (3) under (bb') in line (1).

Now let us call the conjugate pair into which (bb') reverts with respect to (aa'), by the symbol (pp'), which therefore we write, within brackets, in line (1), over bb' in line (2); and also in line (2) under bb' in line (1).

Then, since the collation (2)-(3) is reciprocal, we may write (pp') in (3) under yy' in (2), and therefore over aa' in (5). For, (2)-(5) being a reversion, aa' comes under yy' just as yy' comes under aa'.

But the collation (3)-(5) may be regarded as a transformation, since it might be effected by two double reversions, viz. (3)-(2)-(1), and (1)-(2)-(5), and it might therefore be determined as the transformation which changes (pp') into (aa'); and it is therefore identical with the collation (1)-(4). We may therefore write (yy') in (4), under xx' in (1), since we have yy' in (5) under xx' in (3). That is, when we transformed (aa') into (bb'), we also changed (xx') into (yy').

It is worth while trying to realise clearly what transformations in the three systems of conjugate pairs effect. In a positive system, if we revert successively about the two pairs forming a quartette, the resultant conjugate pairs are all the same as initially, but all reflected. This is equivalent to a transformation which passes each conjugate pair in review all round the whole group of conjugate pairs, though a second similar transformation would be required to transform each unit all round the whole group of units, so that we may say such a transformation passes an unit only halfway round the group of units. Similarly we may say that a transformation which changes a conjugate pair into the pair which forms with it a quartette is half the transformation round the whole group of conjugate pairs. We can go on in this way defining transformations forming any finite fraction of a complete cycle; and of the transformations so defined we may say that a finite number of repetitions of any one of them will transform a pair once round the group or more, but that it may be repeated as often as we please, if we may repeat the cycle over and over again.

In the cases of negative or zero systems, however, we have no means of dividing up the whole group, whether of conjugate pairs or of units, into a finite number of similar parts by transformations. It is impossible to name any transformation which, repeated a finite number of times, would pass either a conjugate pair or a single unit all round the group. Whatever transformation we choose, we could only say that *in the limit*, if it were to be repeated an infinite number of times, we might regard it

as transforming a conjugate pair from one self-conjugate unit to the other in a negative system, or from the self-conjugate unit back to itself in a zero system. And even so it will be noted that in a negative system an *unit* can only be transformed in one of the two parts into which the self-conjugate boundary divides the whole group; not even an infinite number of reversions will ever begin to get it past that boundary.

The geometrical analogue is obvious, but we must be on our guard as to one point. In Geometry we say that the straight line in a Riemann's space is closed and finite in length, that is, we could walk round in a finite number of steps; but in saying this we tacitly assume that they are "finite" steps, which means finite in relation to our space. If we only meant that the steps were finite in relation to our own legs, there would be no logical ground for saying that a finite number of them would take us all round a straight line. And similarly, when we say that an Euclidean straight line is infinite, we cannot logically infer that it is not closed, nor that it is unbounded. Nor does the fact that we cannot transform all the way round a negative system of conjugate pairs preclude our considering it as a whole, from the point of view of Order, just as well as a positive system. There is no sense in saying that we could not pass all its units or conjugate pairs in review in any finite time; for in any case we could not really do that with any continuous group. Moreover, we have nothing to do with the time taken up in passing in review; that it takes any time at all is an irrelevant accident, due to the constitution of the human mind, and not to the Order of the groups we are contemplating. I say this in order to point out that the fact that, in Euclidean and hyperbolic geometries, as generally expounded, there are no conjugate pairs, must not be assumed to be any refutation of the proposition with which I started; though the point cannot be completely cleared up without discussing the cataloguing of groups of higher orders than the first, which would be beyond the scope of the present papers.

(*To be continued.*)

MATHEMATICAL NOTES.

369. [A. 1. c.] *Note on* $1^m+2^m+3^m+\ldots+n^m$.

Take for instance $m=5$.

We have
$$(n+\tfrac{1}{2})^6-(n-\tfrac{1}{2})^6=6n^5+5n^3+\tfrac{5}{8}n,$$
$$\tfrac{5}{4}(n+\tfrac{1}{2})^4-\tfrac{5}{4}(n-\tfrac{1}{2})^4=5n^3+\tfrac{5}{4}n,$$
$$\tfrac{7}{16}(n+\tfrac{1}{2})^2-\tfrac{7}{16}(n-\tfrac{1}{2})^2=\tfrac{7}{8}n.$$

Thus
$$n^5=\phi(n+\tfrac{1}{2})-\phi(n-\tfrac{1}{2}),$$
where $\phi(x)$ stands for $\frac{1}{6}x^6-\frac{5}{24}x^4+\frac{7}{96}x^2-\frac{1}{128}$, so that $\phi(\frac{1}{2})=0$.

It follows that
$$1^5+2^5+\ldots+n^5=\phi(n+\tfrac{1}{2}).$$

Similarly, when m has any *odd* positive value, $1^m+\ldots+n^m$ is equal to an *even* polynomial in $n+\frac{1}{2}$ of degree $m+1$, say $\phi(n+\frac{1}{2})$.

When m is *even* we have in like manner
$$n^m=\phi(n+\tfrac{1}{2})-\phi(n-\tfrac{1}{2}),$$
where ϕ is an *odd* polynomial. Thus
$$0^m=\phi(\tfrac{1}{2})-\phi(-\tfrac{1}{2})=2\phi(\tfrac{1}{2}) \text{ and } \phi(\tfrac{1}{2})=0,$$
so that *whether* m *is odd or even* $1^m+2^m+\ldots+n^m$ *is equal to an even or odd polynomial* $\phi(n+\frac{1}{2})$, *of degree* $m+1$, *and* $\phi(\frac{1}{2})=0$. The leading coefficient

in this polynomial is clearly $\frac{1}{m+1}$, and hence when it is arranged in powers of n (instead of $n+\frac{1}{2}$) the coefficient of n^m is the same as in

$$\frac{1}{m+1}(n+\tfrac{1}{2})^{m+1}, \text{ that is, } \tfrac{1}{2}.$$

Again,
$$n^m = \phi(n+\tfrac{1}{2}) - \phi(n-\tfrac{1}{2})$$
$$= \phi(n+\tfrac{1}{2}) + (-1)^m \phi(-n+\tfrac{1}{2}),$$

and thus *when* $\phi(n+\frac{1}{2})$ *is arranged in powers of* n *the powers* n^{m-2}, n^{m-4}, ... *are all wanting*, as well as the absolute term.

If we put $S_m(x)$ for $\phi(x+\frac{1}{2}) \div m!$ we have $S_m(x) - S_m(x-1) = \frac{x^m}{m!}$, and thus $S'_m(x) - S'_m(x-1) = \frac{x^{m-1}}{(m-1)!}$ by differentiating.

Hence, $S_{m-1}(x) = S'_m(x) - S'_m(0)$, $S_m(x) = xS'_m(0) + \int_0^x S_{m-1}(x)dx$.

Thus, $S_2(x)$, $S_3(x)$ ... can be found by the following rule—when m is odd $S_m(x)$ is the integral of $S_{m-1}(x)$ and vanishes with x: when m is even $S_m(x)$ is the integral of $S_{m-1}(x)$, vanishing with x, plus such a multiple of x that the whole may vanish when $x = -1$.

If we begin with $S_1(x) = \frac{1}{2}x^2 + \frac{1}{2}x$, it is not hard to prove by induction that the polynomials S_m, as formed successively by the rule now stated, have the properties that have been mentioned.

$S_{2m+1}(x)$ contains the factor x^2, and therefore also $(x+1)^2$,

$S_{2m}(x)$ contains the factors $x(x+\frac{1}{2})$, $x+1$.

If we write as usual

$$S_m(x) = \frac{x^{m+1}}{(m+1)!} + \frac{1}{2}\frac{x^m}{m!} + \frac{B_1 x^{m-1}}{2!(m-1)!} - \frac{B_2 x^{m-3}}{4!(m-3)!} + \dots$$

and
$$\Sigma_m(x) = \frac{1}{2^m} S_m(2x) - S_m(x),$$

we have
$$\Sigma_m(x) - \Sigma_m(x-1) = \frac{1}{2^m m!}\{(2x)^m + (2x-1)^m\} - \frac{x^m}{m!}$$
$$= (x-\tfrac{1}{2})^m / m!.$$
$$= \{\phi(x) - \phi(x-1)\}/m!.$$

Thus,
$$\phi(x) = m!\{\Sigma_m(x) - \Sigma_m(\tfrac{1}{2})\}$$

and
$$\frac{\phi(x)}{m!} = \frac{x^{m+1}}{(m+1)!} - \left(1 - \frac{1}{2}\right)\frac{B_1 x^{m-1}}{2!(m-1)!} + \left(1 - \frac{1}{2^3}\right)\frac{B_3 x^{m-3}}{4!(m-3)!} - \dots,$$

the coefficients following the law thus indicated except in the absolute term, which is so fixed that $\phi(\frac{1}{2}) = 0$.

Hence the coefficients in $\phi(x)$ as well as in $\phi(x \pm \frac{1}{2})$ can be simply expressed by means of Bernoulli's numbers: also the sum of the m^{th} powers of the first n odd numbers is
$$2^m\{\phi(n) - \phi(0)\}.$$
A. C. DIXON.

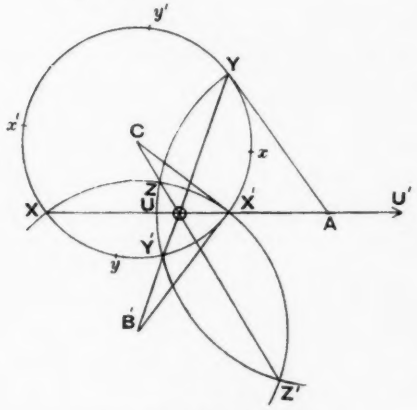
370. [K. 11. a.] *Three circles mutually orthogonal.*

The following theorems have been considered hitherto in relation to a single triangle. It will be found that the proofs are much simplified by considering a system of three circles A, B, C mutually orthogonal.

The figure, from its symmetry, gives three examples of each theorem.

1. Since YOY' is the polar of A, $(XOX'A)$ is harmonic ;

$$\therefore \sin OYX : \sin OYX' = \sin AYX : \sin AYX'$$
$$= \sin YX'X : \sin YXX'$$
$$= YX : YX' ;$$
$$\therefore XO : X'O = XY \cdot OY \cdot \sin OYX : X'Y \cdot OY \cdot \sin OYX'$$
$$= YX^2 : YX'^2.$$



Similarly, $$XO : X'O = Y'X^2 : Y'X'^2 ;$$

and $$YO : Y'O = XY^2 : XY'^2 = X'Y^2 : X'Y'^2.$$

2. If p_1, p_2, p_3, p_4 be $\perp^{rs}$ from O on XY, YX', $X'Y'$ and $Y'X$,

$$p_1 : p_2 = \sin OYX : \sin OYX' = YX : YX' ;$$

$$\therefore \frac{p_1}{YX} = \frac{p_2}{YX'} = (\text{similarly}) \frac{p_3}{X'Y'} = \frac{p_4}{Y'X}.$$

3. Since $(XOX'A)$ is harmonic, so also is the range $Y(XOX'A)$.
Hence YY' bisects all chords of the $\triangle$ YXX' which are $\parallel$ to the tangent at Y, or $\perp^{r}$ to CY.

4. The circle A cutting the circle C orthogonally,
UY bisects angle XYX', and UY' bisects $XY'X'$;

$$\therefore XY : X'Y = XU : X'U = Y'X : Y'X' ;$$
$$\therefore XY \cdot X'Y' = X'Y \cdot Y'X = \tfrac{1}{2} \cdot XX' \cdot YY'.$$

5. Invert the circle C into itself with centre Z, $XYX'Y'$ being inverted into $xyx'y'$.

Then $$yx = YX \cdot \frac{k^2}{ZX \cdot ZY} : yx' = YX' \cdot \frac{k^2}{ZX' \cdot ZY'}$$

But $$YX : YX' = XU : X'U = ZX : ZX' ;$$

$$\therefore yx = yx', \text{ etc.}$$

Hence $xyx'y'$ has all its sides equal, and is therefore a *square*.
So $XYX'Y'$ can be inverted into another square, with Z' for pole.

6. Taking YXX' as $\triangle$ of reference, draw $Zd \cdot Ze \cdot Zf$ $\perp^{rs}$ to XX', YX', XY.
Then in the pedal $\triangle$ def,

$$\sin e : \sin f = df : de = ZX \cdot X'Y : ZX' \cdot XY.$$

But $$ZX ; ZX' = XY ; X'Y ; \quad \therefore e = f,$$

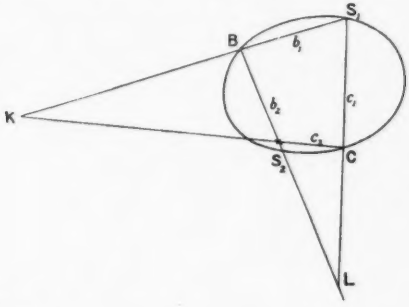
Again, $Y+d=XZX'=\pi-XX'C=\frac{1}{2}\pi+Y$; $\therefore\ d=\frac{1}{2}\pi$,
so that the $\triangle\ yxx'$, known to be similar to *def*, has angles $\frac{1}{2}\pi$, $\frac{1}{4}\pi$, $\frac{1}{4}\pi$:
so for the other triangles of $xy.x'y'$; another proof that it is a square.

7. Note that A is the orthocentre of BOC; and the circle A is the polar circle. W. GALLATLY.

371. [L¹. 1. d.] *New Proof of Homographic Property of a Conic from Pole and Polar.*

Let S_1, B, S_2, C be four points on a conic.

Then $\left.\begin{matrix} b_2c_1, \\ b_1c_2, \end{matrix}\right\}$ *i.e.* KL, passes through pole of S_1S_2, and taking any pt. P on conic, $\left.\begin{matrix} p_1c_2, \\ p_2c_1 \end{matrix}\right\}$ passes through the pole of S_1S_2, etc.



Call this pole H.

Then
$$H\{p_1b_2,\ p_1'b_2,\ p_1''b_2 \ldots\}$$
$$=H\{p_2b_1,\ p_2'b_1,\ p_2''b_1 \ldots\}.$$

$\therefore$ ranges are in perspective,

i.e.
$$S_1\{p_1,\ p_1',\ p_1'' \ldots\}$$
$$=S_2\{p_2,\ p_2',\ p_2'' \ldots\}.$$

S. ANDRADE.

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